Estimation of Hamiltonian Parameters from Thermal States

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We upper bound and lower bound the optimal precision with which one can estimate an unknown Hamiltonian parameter via measurements of Gibbs thermal states with a known temperature. The bounds depend on the uncertainty in the Hamiltonian term that contains the parameter and on the term's degree of noncommutativity with the full Hamiltonian: higher uncertainty and commuting operators lead to better precision. We apply the bounds to show that there exist entangled thermal states such that the parameter can be estimated with an error that decreases faster than $1/\sqrt{n}$, beating the standard quantum limit. This result governs Hamiltonians where an unknown scalar parameter (e.g., a component of a magnetic field) is coupled locally and identically to n qubit sensors. In the high-temperature regime, our bounds allow for pinpointing the optimal estimation error, up to a constant prefactor. Our bounds generalize to joint estimations of multiple parameters. In this setting, we recover the high-temperature sample scaling derived previously via techniques based on quantum state discrimination and coding theory. In an application, we show that noncommuting conserved quantities hinder the estimation of chemical potentials.

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Substantial work has been devoted to determining the precision with which a Hamiltonian parameter can be estimated from measurements on a time-evolving system. For instance, consider a spin network immersed in a magnetic field μ. The network's state acquires information about the field's magnitude. Measuring copies of the state can reveal μ. The quantum Cramér-Rao bound sets an asymptotically saturable lower bound on the precision with which the parameter can be estimated [1,2]. In this dynamical setting, squeezed states of light and entangled states can serve as metrological resources [3,4].

Here, we focus on the less explored problem of estimating parameters from systems in a thermal state

$$\rho = \frac{1}{Z_\beta} e^{-\beta H} = \frac{1}{Z_\beta} \sum_j e^{-\beta \omega_j} |j\rangle\langle j| \tag{1}$$

at a known inverse temperature β. ω_j and $|j\rangle$ are the Hamiltonian's eigenvalues and eigenvectors, $H = \sum_j \omega_j |j\rangle\langle j|$, and $Z_\beta := \mathrm{Tr}(e^{-\beta H})$ is the partition function. The parameters of H could be unknown. Thermalization to ρ is the typical outcome of interactions with environments [5,6]. Probing the environment could yield information about β [7]. The system could also thermalize to ρ, in a range of platforms including superconducting qubits and neutral atoms, through state-preparation protocols [8–10].

The thermal state encodes information about the Hamiltonian parameters. In this Letter, we explore the extent to which thermal states constitute good metrological resources for estimating Hamiltonian parameters. This approach complements the literature devoted to the metrological power of time-evolving quantum states.

We consider M-term Hamiltonians:

$$H = \sum_{l=1}^{M} H_l = \sum_{l=1}^{M} \mu_l A_l. \tag{2}$$

The A_l are Hermitian operators, and the μ_l are real coefficients. The μ_l could represent local or global fields or coupling constants (Fig. 1). We bound the precision with which the μ_l can be estimated from measurements of copies of ρ. To achieve this goal, we will use the multiparameter quantum Cramér-Rao bound, which constrains the estimation of a set of parameters [11].

The quantum Cramér-Rao bound relates the minimum estimation error to the quantum Fisher information [12]. The quantum Cramér-Rao bound has been applied, for example, to the field of thermometry [7,13–19]. The bound implies the minimum uncertainty with which a temperature

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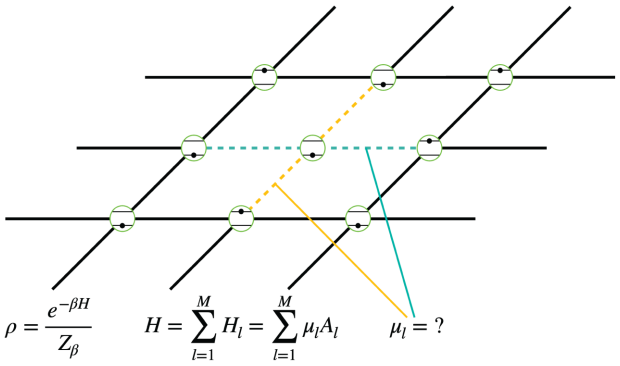

FIG. 1. Estimating Hamiltonian parameters from thermal states. How accurately can one determine μ_l, which can be a coupling constant—pictured here as yellow and teal dashed lines for a system of qubits on a lattice—or a field, in a Hamiltonian H from measurements performed on $\mathcal{N}$ copies of a thermal state $\rho = e^{-\beta H}/Z_\beta$? We use the quantum Cramér-Rao bound to derive saturable upper and lower bounds on the optimal precision with which such Hamiltonian parameters can be estimated.

T can be estimated from $\mathcal{N}$ measurements: $\mathrm{var}(\hat{T}_{\mathrm{opt}}) = T^4\{1/[\mathcal{N}(\Delta H)^2]\}$, where $(\Delta H)^2 \coloneqq \langle H^2\rangle - \langle H\rangle^2$ is the Hamiltonian's variance in the thermal state [11]. Whenever x denotes a parameter to be estimated, we mean by $\hat{x}$ an estimator. Higher energy variances allow for better parameter estimation. This result echoes the relative error $\mathrm{var}(\hat{\mu}_{\mathrm{opt}})/\mu^2 = \{1/[4\mathcal{N}t^2(\Delta H)^2]\}$ with which a global parameter μ can be estimated from measurements of copies of a pure state evolving under the Hamiltonian $H = \mu A$ for a time t. In related work, Refs. [20,21] geometrically characterize the Fisher metric to study the role of phase transitions in thermometry. This Letter focuses on the error in estimates of an *arbitrary* Hamiltonian parameter, rather than the error in temperature estimation.

Several studies have concerned the reconstruction of a Hamiltonian from its eigenstates [22–28], from steady states [29], or from Gibbs states [28,30]. Recent results under the umbrella of the "Hamiltonian-learning problem" provide algorithms for estimating Hamiltonian parameters while minimizing (i) the number of copies of the thermal state ρ needed (the sample complexity) and (ii) the algorithm's runtime (the time complexity) [31–34]. Such complexity-theoretic approaches focus on (a) the asymptotic sample and time complexities' dependence on β and (b) the number of unknown parameters. In contrast, we leverage the quantum Cramér-Rao bound to identify how the uncertainties in the A_l's, and the noncommutativity of the A_l's with the thermal state, influence the minimum precision with which the μ_l can be estimated. Upon pinpointing the uncertainties' influence on precision, we can construct a many-body model that beats the standard quantum limit.

This Letter is organized as follows. First, we review the quantum Fisher information, a powerful tool for analyzing parameter estimation. We bound the quantum Fisher information obtainable about one Hamiltonian parameter, then bound the precision with which the parameter can be estimated. These bounds enable us to identify a many-body model in which the achievable precision beats the standard quantum limit. Extending beyond one Hamiltonian parameter, we then bound the precision with which multiple parameters can be estimated simultaneously. Finally, we discover that noncommutation of conserved quantities (*charges*) hinders the estimation of chemical potentials. Noncommuting charges are particularly quantum (due to the importance of noncommutation in quantum measurement disturbance, Heisenberg uncertainty, etc.) and have been of recent thermodynamic interest [35].

The quantum Fisher information matrix—The multiparameter quantum Cramér-Rao bound constrains the statistics of any estimator $\hat{\vec{\mu}}$ of the parameters μ_l [11]:

$$\mathrm{cov}(\hat{\vec{\mu}}) \geq \frac{1}{\mathcal{N}}\mathcal{F}^{-1}. \tag{3}$$

$\mathcal{N}$ denotes the number of experimental repetitions. $\mathcal{F}$ denotes the quantum Fisher information matrix, with components

$$\mathcal{F}_{lm} \coloneqq 2\sum_{jk}\frac{\mathrm{Re}[\langle j|\partial_l\rho|k\rangle\langle k|\partial_m\rho|j\rangle]}{p_j + p_k}. \tag{4}$$

The state eigendecomposes as $\rho = \sum_j p_j|j\rangle\langle j|$. Thus, the quantum Fisher information matrix characterizes the precision with which parameters μ_l can be estimated jointly. The multiparameter quantum Cramér-Rao bound is saturated when the optimal measurements for estimating the μ_l are compatible. Mathematically, this condition is met if and only if $\mathrm{Tr}(\rho[L_l, L_m]) = 0$. The symmetric logarithmic derivative L_l is implicitly defined by $\partial_l\rho =: \frac{1}{2}\{\rho, L_l\}$ [11]. Throughout this work, we denote partial derivatives by $\partial_l \coloneqq (\partial/\partial\mu_l)$.

The diagonal matrix element $\mathcal{F}_{ll}$ quantifies the minimum precision with which one unknown μ_l can be estimated if all other parameters are known. The single-parameter quantum Cramér-Rao bound says that every estimator $\hat{\mu}_l$ has a variance

$$\mathrm{var}(\hat{\mu}_l) \geq \frac{1}{\mathcal{N}\mathcal{F}_{ll}}. \tag{5}$$

Optimized measurements saturate this bound [2,12]. Equations (3) and (5) thus pinpoint the quantum Fisher information as a powerful tool that determines ultimate limits on quantum metrology. The stronger ρ's dependency on μ_l, the higher the quantum Fisher information $\mathcal{F}_{ll}$ [Eq. (4)], and so the greater the precision.

Bounds on the quantum Fisher information—Exactly evaluating the quantum Fisher information in Eq. (4) can be

difficult, requiring knowledge of the Hamiltonian's spectrum and eigenstates. Therefore, it is desirable to bound $\mathcal{F}_{ll}$ in terms of more-easily-calculable quantities. We derive two sets of upper and lower bounds on the quantum Fisher information of the μ_l in Eq. (2):

$$\mathcal{F}_{ll} \leq \beta^2 (\Delta A_l)^2 \tag{6a}$$

$$\mathcal{F}_{ll} \geq 4\beta^2 c_1 (\Delta A_l)^2, \tag{6b}$$

and

$$\mathcal{F}_{ll} \leq 2.4 c_2 \beta^2 \left((\Delta A_l)^2 - \frac{1}{2} \| [\sqrt{\rho}, A_l] \|_2^2 \right) \quad \text{and} \tag{7a}$$

$$\mathcal{F}_{ll} \geq 0.8\beta^2 \left((\Delta A_l)^2 - \frac{1}{2} \| [\sqrt{\rho}, A_l] \|_2^2 \right). \tag{7b}$$

$\Delta A_l = \sqrt{\langle A_l^2 \rangle - \langle A_l \rangle^2}$ is the uncertainty of operator A_l in ρ; $\|A\|_2^2 \coloneqq \mathrm{Tr}(AA^\dagger)$; and we have defined

$$c_1 \coloneqq \tanh^2(\beta \|H\|_s/2)/(\beta \|H\|_s)^2 \quad \text{and} \tag{8a}$$

$$c_2 \coloneqq 2c_1 \cosh(\beta \|H\|_s/2). \tag{8b}$$

The $\|H\|_s \coloneqq \max_j \omega_j - \min_j \omega_j$ is the Hamiltonian seminorm defined by the maximum energy gap. We derive the bounds by computing the thermal state's quantum Fisher information, then algebraically manipulating the expression (Appendix II in the Supplemental Material [36]).

Equation (6) constrains the quantum Fisher information about μ_l in terms of ΔA_l, resembling expressions for the quantum Fisher information about β in thermometry [11]. Equation (7) constrains the quantum Fisher information about μ_l also in terms of the Wigner-Yanase skew information $\frac{1}{2} \| [\sqrt{\rho}, A_l] \|_2^2$. The skew information was proposed as a means to discriminate quantum and classical contributions to uncertainty [46,47]. It has found applications in parameter estimation [15,48,49], as an asymmetry measure [50], and as a coherence measure [51,52]. The difference $(\Delta A_l)^2 - \frac{1}{2} \| [\sqrt{\rho}, A_l] \|_2^2$ signifies the classical uncertainty about A_l [47]. This classical uncertainty vanishes for pure states. We emphasize that the bounds in Eqs. (6) and (7) are mathematically distinct—neither is tighter than the other in all regimes.

When the temperature is high relative to the maximum energy gap ($\beta \|H\|_s \ll 1$), $c_1 \approx c_2/2 \approx 1/4$. The upper and lower bounds in Eq. (6) coincide, while the upper and lower bounds in Eq. (7) differ by a prefactor of 1.2. That is, our bounds are saturated, up to a constant prefactor, at high temperatures. Our bounds pinpoint $\mathcal{F}_{ll}$ by tightly sandwiching it.

The upper bound (7a) is also saturable, up to a constant prefactor, at low temperatures. To show this, we denote by μ the magnitude of a field $\mu\sigma_z$ acting on a qubit with a Hamiltonian $H = \Omega_x \sigma_x + \Omega_z \sigma_z + \mu\sigma_z$. The σ_α's are Pauli matrices. The quantum Fisher information and its upper bound (7a) can be calculated exactly. At low temperatures ($\beta \|H\|_s \gg 1$), $\mathcal{F}_\mu \approx 16\Omega_x^2/\|H\|_s^4 \leq 2.4 c_2 \beta^2 \times ((\Delta A_l)^2 - \frac{1}{2} \| [\sqrt{\rho}, A_l] \|_2^2) \approx 19.2\Omega_x^2/\|H\|_s^4$; and, at high temperatures ($\beta \|H\|_s \ll 1$), $\mathcal{F}_\mu \approx \beta^2 \leq 2.4 c_2 \beta^2 ((\Delta A_l)^2 - \frac{1}{2} \| [\sqrt{\rho}, A_l] \|_2^2) \approx 1.2\beta^2$ (Appendix IV in the Supplemental Material [36]). The contribution of the Wigner-Yanase skew information is necessary for obtaining a saturable bound at low temperatures.

Reference [15] contains the closest previous result: $\mathcal{F}_{ll} \leq \beta^2 \int_0^1 \mathrm{Tr}(\rho^a \delta A_l \rho^{1-a} \delta A_l) da$, with $\delta A_l \coloneqq A_l - \langle A_l \rangle$. Yet our upper bounds (6a), (7a), and the bound in Ref. [15] are different: again, no bound is tighter than another in all regimes. To our knowledge, Eqs. (6b) and (7b) are the first lower bounds on thermal states' quantum Fisher information. We demonstrate this explicitly using a spin-chain example in Appendix V of the Supplemental Material [36].

Bounds on single-parameter estimation errors—Consider estimating an unknown parameter μ_l. We denote the *optimal* error by $\sqrt{\mathrm{var}_{\mathrm{opt}}(\hat{\mu}_l)}$. The single-parameter quantum Cramér-Rao bound (5) is saturable by suitably chosen estimators [12]. Therefore, Eqs. (6) and (7) engender two sets of upper and lower bounds on $\sqrt{\mathrm{var}_{\mathrm{opt}}(\hat{\mu}_l)}$. The relative error $[\sqrt{\mathrm{var}_{\mathrm{opt}}(\hat{\mu}_l)}/|\mu_l|]$ achievable with $\mathcal{N}$ copies of a thermal state is

$$\frac{1}{\beta\sqrt{\mathcal{N}}\Delta H_l} \leq \frac{\sqrt{\mathrm{var}_{\mathrm{opt}}(\hat{\mu}_l)}}{|\mu_l|} \leq \frac{1}{2\beta c_1^{1/2}\sqrt{\mathcal{N}}\Delta H_l}, \tag{9}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{2.4 c_2}\beta\sqrt{\mathcal{N}}((\Delta H_l)^2 - \frac{1}{2}\| [\sqrt{\rho}, H_l] \|_2^2)^{1/2}} \\ &\quad \leq \frac{\sqrt{\mathrm{var}_{\mathrm{opt}}(\hat{\mu}_l)}}{|\mu_l|} \\ &\quad \leq \frac{1}{\sqrt{0.8}\beta\sqrt{\mathcal{N}}((\Delta H_l)^2 - \frac{1}{2}\| [\sqrt{\rho}, H_l] \|_2^2)^{1/2}}. \end{aligned} \tag{10}$$

By Eq. (9), a higher uncertainty ΔH_l in $H_l = \mu_l A_l$ can enable better precision. Meanwhile, Eq. (10) constrains the relative error in terms of the classical uncertainty in $H_l = \mu_l A_l$. Equation (10) also reveals the role of noncommutativity: when A_l does not commute with ρ, the ability to estimate μ_l diminishes. This fact has an analogue in single-parameter estimation in unitary quantum metrology, as detailed in Appendix III of the Supplemental Material [36]. There, μ_l can be encoded in a probe state via Hamiltonian evolution under $H = \mu_l A_l + H'$, for an arbitrary Hermitian H'. If $[A_l, H] \neq 0$—and so

$[\rho, H] \neq 0$ for thermal states ρ—the ability to measure μ_l is diminished [53].

In quantum metrology, the estimation error's scaling with a sensor's size can constitute an entanglement advantage. Consider a system of n subsystems and H_l a sum of n local terms. Superextensive variances $(\Delta H_l)^2 \sim n^\alpha$, with $\alpha > 1$, are atypical for thermal states of spatially local Hamiltonians. For instance, $(\Delta H_l)^2 \sim n$ for states with exponentially decaying correlations [54,55]. From Eq. (9), one would expect the optimal estimation error to scale as $1/(\beta\sqrt{\mathcal{N}}\sqrt{n})$, as in the standard quantum limit [56,57]. This conclusion is consistent with Ref. [58], which implies that the standard quantum limit cannot be beaten with measurements of locally gapped Hamiltonian ground states. At critical points, however, $(\Delta H_l)^2 \sim n$ may be violated [21,59,60]. We can observe violations also with certain nonlocal Hamiltonians.

We now show that one can beat the standard quantum limit in Hamiltonian metrology using thermal states. Consider estimating a field μ by measuring copies of a thermal state of the n-qubit Hamiltonian $H = \mu \sum_{j=1}^n (\sigma_z^j + 1) - \lambda \otimes_{j=1}^n n\sigma_x^j \equiv H_\mu + H_\lambda$. We assume $\lambda > 0$ and $\mu > 0$. Let $|\bar{0}\rangle$ denote the n-fold tensor product of the eigenvalue-(-1) eigenstate of σ_z; and $|\bar{1}\rangle$, the product of the eigenvalue-1 eigenstate. The n-qubit Greenberger–Horne–Zeilinger (GHZ) state $|\Phi\rangle := (|\bar{0}\rangle + |\bar{1}\rangle)/\sqrt{2}$ is a ground state of H_λ. We prove in Appendix VI of the Supplemental Material [36] that $|\Phi\rangle$ is the unique ground state if H_μ is a perturbation ($\mu/\lambda \ll 1$). The variance of H_μ in $|\Phi\rangle$ is $\langle\Phi|H_\mu^2|\Phi\rangle - \langle\Phi|H_\mu|\Phi\rangle^2 = \mu^2 n^2$. Therefore, one might expect that $\Delta H_\mu \sim \mu n^\alpha$, with $\alpha > 1/2$, in low-temperature thermal states. In Appendix IV of the Supplemental Material [36], we prove this expectation, showing that $\alpha = 1$ for $\beta\lambda n \gg 1$. Note this proof does not require that $\mu/\lambda \ll 1$. By Eqs. (9) and (10), this result suggests a minimum relative estimation error that decreases faster than the standard quantum limit $1/\sqrt{n}$. Figure 2 supports this argument, exhibiting a regime with optimal relative estimation errors below $1/\sqrt{n}$. These results would have been difficult to deduce from the expression (4) for the quantum Fisher information. By leveraging our bounds, we found a model that beats the standard quantum limit. While we have not found a metrological advantage over, e.g., GHZ states, thermal states are more prevalent, and we find it insightful to learn that they can serve as metrological resources.

Bounds on multiparameter estimation errors—The single-parameter bounds above apply when all parameters except the target parameter are known. However, our results imply bounds on the error in joint estimates of M Hamiltonian parameters. The variances' sum serves as the error measure. We aim for a total error $\sum_{l=1}^M \mathrm{var}(\hat{\mu}_l) = \epsilon_{\mathrm{err}}^2$. By the multiparameter Cramér-Rao bound (3),

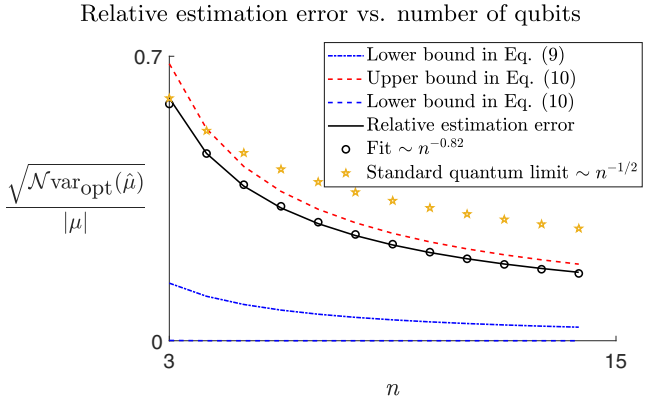

FIG. 2. Beating the standard quantum limit. The figure shows the relative estimation error $\sqrt{\mathrm{var}_{\mathrm{opt}}(\hat{\mu})}/|\mu|$ for the parameter μ in the n-qubit $H = \mu \sum_j^n (\sigma_z^j + 1) - \lambda \otimes_j^n n\sigma_x^j := H_\mu + H_\lambda$. The bounds appear in Eqs. (9) and (10) [the upper bound in (9) is, here, too loose to appear in the plotted range]. We take $\lambda\beta = 2\mu\beta = 6$. As we show in Appendix VI of the Supplemental Material [36], $\Delta H_\mu \sim \mu n$, for large $\beta\lambda n$. A consequence, suggested by Eqs. (9) and (10), is an optimal estimation error that decays faster than $1/\sqrt{n}$.

$$\epsilon_{\mathrm{err}}^2 = \sum_{l=1}^M \mathrm{var}(\hat{\mu}_l) \geq \frac{1}{\mathcal{N}} \mathrm{Tr}(\mathcal{F}^{-1}) \geq \frac{1}{\mathcal{N}} \sum_{l=1}^M \frac{1}{\mathcal{F}_{ll}}. \quad (11)$$

The final inequality holds under the condition $\mathcal{F} > 0$, satisfied if one can estimate every linear combination of parameters [11]. The second inequality is useful for large M, when calculating $\mathcal{F}^{-1}$ is computationally hard.

The second inequality is saturated if and only if $\mathcal{F}$ is diagonal. The first inequality is saturated if and only if $\mathrm{Tr}(\rho[L_l, L_m]) = 0$ [11,37], which occurs when

$$\sum_{\omega_j \neq \omega_k} \frac{(p_j - p_k)^3}{(\omega_j - \omega_k)^2 (p_j + p_k)^2} \langle j|A_l|k\rangle\langle k|A_m|j\rangle = 0 \quad (12)$$

for all $\{l, m\}$, where $p_j = e^{-\beta\omega_j}/Z_\beta$. These are rather stringent conditions violated by typical many-body Hamiltonians.

By combining Eq. (11) with the inequalities (6a) and (7a), we bound the error in the estimation of multiple Hamiltonian parameters. To learn M Hamiltonian parameters with an error ϵ_{err}, one needs a number $\mathcal{N}$ of measurements satisfying

$$\mathcal{N} \geq \frac{1}{\beta^2 \epsilon_{\mathrm{err}}^2} \sum_{l=1}^M \frac{c_2^{-1}/2}{(\Delta A_l)^2 - \frac{1}{2}\|[\sqrt{\rho}, A_l]\|_2^2} \quad \text{and} \quad (13a)$$

$$\mathcal{N} \geq \frac{1}{\beta^2 \epsilon_{\mathrm{err}}^2} \sum_{l=1}^M \frac{1}{(\Delta A_l)^2}. \quad (13b)$$

Consequently,

$$\mathcal{N} = \Omega\left(\frac{M}{\beta^2 \epsilon_{\text{err}}^2} \min_l \frac{1}{(\Delta A_l)^2}\right). \tag{14}$$

We can compare Eq. (14) to complexity-theoretic results [31,32] about the number $\mathcal{N}$ of copies of the state required to learn M Hamiltonian parameters to within an l_2-distance error ϵ defined through $\epsilon^2 = \sum_{l=1}^{M} (\hat{\mu}_l - \mu_l)^2$. At least $\mathcal{N} = \Omega\{[\exp(\beta)M]/[\beta^2\epsilon^2]\}$ samples are required for a Hamiltonian with M terms [32]. At low temperatures, their bound is tighter, as a function of β. Moreover, we have only proven Eq. (13) to be saturable under stringent conditions on the operators A_l. They prove a stronger result: $\mathcal{N} = \mathcal{O}[(M/\beta^2\epsilon^2)\ln(M/\delta)]$ samples suffice to learn the parameters with a constant failure probability δ. In contrast, our results are more general since they concern the average error in estimations of parameters in arbitrary Hamiltonians. Also, our results reveal the roles of uncertainties ΔA_l, and of the state's noncommutativity with A_l, in the estimation error. We compare this Letter's bounds with previous bounds in detail in Appendix VIII (see Table I) of the Supplemental Material [36].

Estimation of chemical potentials—In the presence of conserved charges Q_l, thermalizing systems reach generalized Gibbs states [61–64]

$$\rho_{\beta,\{\mu_l\}} = e^{-\beta\left(H_0 + \sum_l \mu_l Q_l\right)} / Z_{\beta,\{\mu_l\}}. \tag{15}$$

H_0 is the system Hamiltonian. The μ_l are the *chemical potentials* corresponding to the charges, which satisfy $[H_0, Q_l] = 0$ for all l.

Our results imply constraints on the minimum error in estimations of the chemical potentials: we identify $H \equiv H_0 + \sum_l \mu_l Q_l$ and $A_l \equiv Q_l$ in Eq. (1). For example, consider estimating one μ_l. Equations (7), with the quantum Cramér-Rao bound's saturability, imply

$$\frac{1}{2.4 c_2 \beta^2 \mathcal{N}((\Delta Q_l)^2 - \frac{1}{2}\|[\sqrt{\rho}, Q_l]\|_2^2)} \leq \text{var}_{\text{opt}}(\hat{\mu}_l) \leq \frac{1}{0.8\beta^2 \mathcal{N}((\Delta Q_l)^2 - \frac{1}{2}\|[\sqrt{\rho}, Q_l]\|_2^2)}. \tag{16}$$

Classically, all charges commute with each other and so with ρ. Quantum charges can defy this expectation: $[Q_l, Q_m] \neq 0$ [62,63,65,66]. For instance, the two-qubit Hamiltonian $H_0 = \sigma_z \otimes \sigma_z$ conserves charges $Q_1 = \sigma_z \otimes \mathbb{1}$ and $Q_2 = \sigma_x \otimes \sigma_x$ that do not commute with each other. This noncommutation prevents charges from commuting with the state: $[Q_l, \sqrt{\rho}] \neq 0$. This lack of equality implies a quantum disadvantage in parameter estimation: charges' noncommutativity hinders the ability to measure chemical potential μ_1. We inferred this hindrance from the bounds in Eq. (10). It is unclear whether the hindrance can be inferred from Eq. (9), which singles out uncertainty, instead of noncommutativity. This observation underscores how the bounds in Eqs. (9) and (10) are inequivalent and provide different lessons in different contexts.

Discussion—Our bounds highlight how estimation error depends on the uncertainty and noncommutativity of the operators defining the Hamiltonian. The noncommutativity engenders a disadvantage, diminishing precision. See Eq. (10) and Appendix III of the Supplemental Material [36] for a comparison with the estimation of parameters from Hamiltonian evolution.

Furthermore, we found that noncommutativity of conserved charges hinders estimations of chemical potentials. This result contrasts with Refs. [66,67], which show that conserved quantities' noncommutativity provides an advantage in quantum transport processes by decreasing entropy production. Our work therefore contributes to the debate about whether noncommuting charges enhance or hinder desirable properties in information-processing and thermodynamic tasks [35,68].

A natural open problem concerns the bounds' saturability. What concrete protocols saturate the bounds? Also, do single-shot or global measurements of $\mathcal{N}$ copies $\rho^{\otimes\mathcal{N}}$ saturate the bounds [69,70]? Moreover, we found a toy model where, using measurements on a thermal state, one can beat the standard quantum limit for the task of estimating (a component of) a field coupled locally to n qubits. Further work is needed to determine whether one can use thermal states of more-physically-realistic, fully local Hamiltonians to beat the standard quantum limit, possibly by exploiting criticality [21,59,60,71,72]. Finally, it would be interesting to explore the implications of our work toward the complexity of learning thermal states [73,74].

Note added—Reference [75], which studies the Hamiltonian learning problem at all temperatures, was posted during the preparation of this manuscript.

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