Supplemental Material for "The Lieb-Robinson light cone for power-law interactions"

(Dated: September 21, 2021)

In this Supplemental Material, we provide a rigorous proof of Theorem 1 in the main text (Sec. S1) and details on the applications of the bound to connected correlators, topologically ordered states, and simulations of local observables (Sec. S2). We also summarize the tight Lieb-Robinson bounds for $\alpha \geq 0$ (Sec. S3) and compare our proof strategy to previous works (Sec. S4).

CONTENTS

S1. PROOF OF THEOREM 1

In this section, we provide a rigorous proof of Theorem 1. We first summarize the lemmas we use in the proof of the theorem, followed by the proofs of the lemmas in Secs. S1 A to S1 C.

For convenience, we first recall the definitions from the main text. We consider a d-dimensional lattice of qubits Λ and, acting on this lattice, a two-body power-law Hamiltonian $H(t)$ with exponent α. Specifically, we assume $H(t) = \sum_{i,j\in\Lambda} h_{ij}(t)$ is a sum of two-body terms h_{ij} supported on sites i, j such that $\|h_{ij}(t)\| \leq 1/\text{dist}(i,j)^\alpha$ for all $i \neq j$, where $\|\cdot\|$ is the operator norm and $\text{dist}(i,j)$ is the distance between i, j. In this paper, we assume $2d < \alpha < 2d+1$.

We use $\mathcal{L}$ to denote the Liouvillian corresponding to the Hamiltonian H, i.e. $\mathcal{L}\left|O\right) \equiv i\left|[H,O]\right)$ for all operators O, and use $e^{\mathcal{L}t}\left|O\right) \equiv \left|O(t)\right)$ to denote the time evolved version of the operator O. Similarly to the main text, we use $\mathbb{P}_r^{(i)}\left|O\right)$ to denote the projection of O onto sites that are at least a distance r from site i. In particular, if i is the origin of the lattice, we may also drop the superscript i and simply write $\mathbb{P}_r$ for brevity.

Given a unit-norm operator O initially supported at the origin, $\mathbb{P}_r e^{\mathcal{L}t}\left|O\right)$ provides the fraction of the time-evolved version of the operator O that is supported at least a distance r from the origin at time t. The identity [S1]

$$\frac{1}{2} \leq \frac{\left\|\mathbb{P}_r e^{\mathcal{L}t}\left|O\right)\right\|}{\sup_A \left\|[A, e^{\mathcal{L}t}O]\right\|} \leq 2, \tag{S1}$$

where the supremum is taken over all unit-norm operators A supported at least a distance r from O, establishes the equivalence between the projector and the unequal-time commutator commonly used in the Lieb-Robinson literature.

Theorem 1. For any $\alpha \in (2d, 2d+1)$ and $\varepsilon \in \left(0, \frac{(\alpha-2d)^2}{(\alpha-2d)^2+\alpha-d}\right)$, there exist constants $c, C_1, C_2 \geq 0$ such that

$$\left\|\mathbb{P}_r e^{\mathcal{L}t}\left|O\right)\right\| \leq C_1\left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\frac{\varepsilon}{2}} + C_2\frac{t}{r^{\alpha-d}} \tag{S2}$$

holds for all $t \leq c r^{\alpha-2d-\varepsilon}$.

Our strategy is to divide the terms of the Hamiltonian by their interaction range and prove a Lieb-Robinson-like bound recursively for each range. Specifically, let $\ell_0 = 0$ and $\ell_k \equiv L^k$ for $k = 1, 2, \ldots, n$, where $L > 1$ to be chosen later,

$$n = \left\lfloor \frac{1}{\log L} \log \left[r \left(\frac{t}{r^{\alpha-2d}} \right)^{\eta} \right] \right\rfloor, \tag{S3}$$

and $\eta \in (0, \frac{1}{\alpha-d})$ is an arbitrary small constant. For our convenience, we set $\ell_{n+1} = r_*$, where r_* is the diameter of the lattice. We then divide the Hamiltonian into $H = \sum_{k=1}^{n+1} V_k$, where $V_k = \sum_{i,j:\ell_{k-1}<\mathrm{dist}(i,j)\le\ell_k} h_{ij}$ consists of terms h_{ij} such that the distance between i, j is between ℓ_{k-1} and ℓ_k. We also use $H_k = \sum_{j=1}^{k} V_k$ to denote the sum of interactions whose lengths are at most ℓ_k and $\mathcal{L}_k = i[H_k, \cdot]$ are the corresponding Liouvillians. Note that $H_{n+1} = H$ contains every interaction of the Hamiltonian.

We start with a standard Lieb-Robinson bound for H_1 [S2, S3], i.e.

$$\left\|\mathbb{P}_r e^{\mathcal{L}_1 t} |O)\right\| \le \exp\left[\frac{v_1 t - r}{\ell_1}\right], \tag{S4}$$

where $v_1 = 4e\tau\ell_1$ is proportional to ℓ_1 and $\tau = \max_i \sum_{j\in\Lambda, j\ne i} 1/\mathrm{dist}(i,j)^\alpha$ is a constant for all $\alpha > d$, and recursively prove bounds for $H_2, H_3, \ldots, H_n$ using the following lemma:

Lemma S1. Suppose for $\ell_k \ge 1$, we have

$$\left\|\mathbb{P}_r e^{\mathcal{L}_k t} |O)\right\| \le \exp\left[\frac{v_k t - r}{\ell_k}\right], \tag{S5}$$

for some unit-norm operator O supported at the origin. Then for $\ell_{k+1} > \ell_k$, we have

$$\left\|\mathbb{P}_r e^{\mathcal{L}_{k+1} t} |O)\right\| \le \exp\left[\frac{v_{k+1} t - r}{\ell_{k+1}}\right]. \tag{S6}$$

where

$$v_{k+1} = \xi \log(r_*) v_k + \nu\lambda \frac{\ell_{k+1}^{2d+1}}{\ell_k^{\alpha}} \tag{S7}$$

and ξ, ν, λ are constants that may depend only on d.

Note that each of the bounds in the series has a logarithmic dependence on the diameter r_* of the lattice. We later show that this dependence on r_* can be replaced by a similar logarithmic dependence on r, leading to a logarithmic correction in the light cone. After applying Lemma S1 $n-1$ times, we arrive at a bound for the evolution under H_n:

$$\left\|\mathbb{P}_r e^{\mathcal{L}_n t} |O)\right\| \le \exp\left[\frac{v_n t - r}{\ell_n}\right], \tag{S8}$$

where

$$v_n = x^{n-1}(v_1 - L^{2d+1}\nu\lambda) + x^{n-1} L^{2d+1}\nu\lambda \left[1 + \frac{L^{2d+1-\alpha}}{x} + \ldots \left(\frac{L^{2d+1-\alpha}}{x}\right)^{n-1}\right] \tag{S9}$$

and $x \equiv \xi \log r_*$. We now choose $L = x^{1/(2d+1-\alpha)}$ so that

$$v_n = x^{n-1}[v_1 + (n-1)L^{2d+1}\nu\lambda]. \tag{S10}$$

At this point, we have a bound for the evolution under H_n, which contains most terms of the Hamiltonian except for those with range larger than ℓ_n. With the value of n in Eq. (S3), we eventually show that the bound Eq. (S8) has the desired light cone $t \gtrsim r/v_n \sim r^{\alpha-2d}$.

Next, we add the remaining long-range interactions in $H - H_n$, i.e. those with range larger than ℓ_n, to the bound. The result is the following lemma, which we prove in Sec. S1 B.

Lemma S2. Given any $\varepsilon > 0$, there exist constants C, c, κ, δ such that

$$\left\|\mathbb{P}_r e^{\mathcal{L} t} |O)\right\| \le C \log^\kappa r_* \left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon} \tag{S11}$$

holds for all $t \le c r^{\alpha-2d-\varepsilon}/\log^\delta r_*$.

The bound at this point still has an undesirable feature: it depends on the size of the lattice r_*. Finally, we show in Sec. S1 C that we can remove this dependence on r_* at the cost of adding additional terms to the bound. The result is Theorem 1 presented in the main text.

A. Proof of Lemma S1

In this section, we prove Lemma S1.

Proof. For simplicity, let $V \equiv V_{k+1} = H_{k+1} - H_k$ in this section. We shall move into the interaction picture of H_k and write the time evolution under H_{k+1} as a product

$$\mathcal{T}\exp\left(-i\int_0^t ds\ H_{k+1}(s)\right) = \mathcal{T}\exp\left(-i\int_0^t ds\ e^{\mathcal{L}_k s}V\right)\cdot\mathcal{T}\exp\left(-i\int_0^t ds\ H_k(s)\right) \tag{S12}$$

of an evolution under H_k, for which Eq. (S5) applies, and an evolution $e^{\mathcal{L}_I t}$ under the $V_I(t) = e^{\mathcal{L}_k t}V$.

We decompose every term h_{ij} in V into a sum of products of two single-site operators $u_i^{(\mu)}$:

$$h_{ij} = \sum_\mu J_{ij}^{(\mu)} u_i^{(\mu)} u_j^{(\mu)}, \tag{S13}$$

where $u_i^{(\mu)}$ have unit norms, $J_{ij}^{(\mu)}$ are nonnegative, and $\sum_\mu J_{ij}^{(\mu)} \leq 1/\mathrm{dist}(i,j)^\alpha$. In doing so, we can reduce the evolution of h_{ij} into the evolutions of single-site operators $u_i^{(\mu)}$:

$$\begin{aligned} e^{\mathcal{L}_k t}h_{ij} &= e^{iH_k t}\left[\sum_\mu J_{ij}^{(\mu)} u_i^{(\mu)} u_j^{(\mu)}\right]e^{-iH_k t} = \sum_\mu J_{ij}^{(\mu)} e^{iH_k t}u_i^{(\mu)}e^{-iH_k t}e^{iH_k t}u_j^{(\mu)}e^{-iH_k t} \\ &= \sum_\mu J_{ij}^{(\mu)}\left[e^{\mathcal{L}_k t}u_i^{(\mu)}\right]\left[e^{\mathcal{L}_k t}u_j^{(\mu)}\right], \end{aligned} \tag{S14}$$

where we have used the fact that $e^{\mathcal{L}_k t}(\cdot) = e^{iH_k t}(\cdot)e^{-iH_k t}$ is a unitary transformation.

We then pick a parameter $R \geq \ell_k$ and divide the lattice around i into shells of width R. Specifically, let $\mathcal{B}_r^{(i)}$ denote the ball of radius r centered on i. Let $\mathcal{S}_r^{(i)} = \mathcal{B}_r^{(i)} \setminus \mathcal{B}_{r-R}^{(i)}$ denote the shell of inner radius $r-R$ and outer radius r centered on i. For each μ, we have

$$e^{\mathcal{L}_k t}u_i^{(\mu)} = \left[(\mathbb{I}-\mathbb{P}_R^{(i)}) + (\mathbb{P}_R^{(i)} - \mathbb{P}_{2R}^{(i)}) + (\mathbb{P}_{2R}^{(i)} - \mathbb{P}_{3R}^{(i)}) + \dots\right]e^{\mathcal{L}_k t}u_i^{(\mu)} \equiv \sum_{q=0}^\infty u_{i,q}^{(\mu)}(t), \tag{S15}$$

where the distance in the subscript of the projectors is with respect to i and $u_{i,q}^{(\mu)}$ is supported on $\mathcal{B}_{(q+1)R}^{(i)}$ for $q = 0,1,2,\dots$.

Using Eq. (S5) and the triangle inequality, we can show that

$$\left\|u_{i,q}^{(\mu)}(t)\right\| \leq \left\|\mathbb{P}_{qR}^{(i)}e^{\mathcal{L}_k t}u_i^{(\mu)}\right\| + \left\|\mathbb{P}_{(q+1)R}^{(i)}e^{\mathcal{L}_k t}u_i^{(\mu)}\right\| \leq \exp\left(\frac{v_k t - qR}{\ell_k}\right) + \exp\left(\frac{v_k t-(q+1)R}{\ell_k}\right). \tag{S16}$$

Choosing $R \geq v_k t$ and $R \geq (1+\varepsilon)\ell_k$ for some positive constant ε, we have

$$\left\|u_{i,q}^{(\mu)}(t)\right\| \leq e^{\frac{-(q-1)R}{\ell_k}} + e^{\frac{-qR}{\ell_k}} \leq e^{-(q-1)(1+\varepsilon)} + e^{-q(1+\varepsilon)} \leq (1+e^{1+\varepsilon})e^{-q(1+\varepsilon)} \tag{S17}$$

for all $q = 0,1,2,\dots$. By combining the two legs of h_{ij} together, we arrive at a decomposition $e^{\mathcal{L}_k t}h_{ij} = \sum_{p,q} w_{i,p;j,q}(t)$, where $w_{i,p;j,q}(t) = \sum_\mu J_{ij}^{(\mu)}u_{i,p}^{(\mu)}(t)u_{j,q}^{(\mu)}(t)$ and

$$\|w_{i,p;j,q}(t)\| \leq \frac{(1+e^{1+\varepsilon})^2}{\mathrm{dist}(i,j)^\alpha}e^{-(p+q)(1+\varepsilon)}. \tag{S18}$$

Next, we divide the lattice into complementary hypercubes of length R. We shall prove that $V_I(t)$ actually consists of exponentially decaying interactions between hypercubes. We shall index the hypercubes by their centers, i.e. $\mathcal{C}_x$ denotes the hypercube center at x. Given x, y as the centers of two hypercubes,

$$\tilde{h}_{xy}(t) \equiv \sum_{\substack{i,j,p,q \\ \mathcal{B}_{(p+1)R}^{(i)}\cap\mathcal{C}_x\neq\varnothing \\ \mathcal{B}_{(q+1)R}^{(j)}\cap\mathcal{C}_y\neq\varnothing}} w_{i,p;j,q}(t) \tag{S19}$$

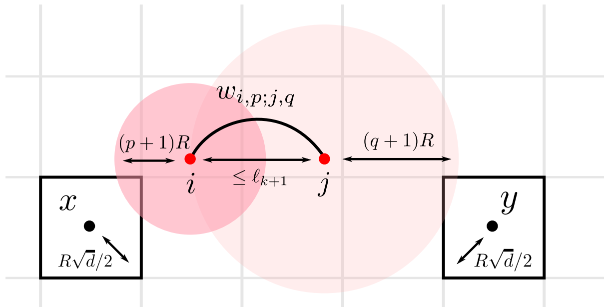

FIG. S1. The effective interaction between two hypercubes $\mathcal{C}_x$ and $\mathcal{C}_y$ comes from the terms $w_{i,p;j,q}$ whose support (the shaded area) overlaps with the cubes.

defines the effective interaction between the cubes $\mathcal{C}_x$ and $\mathcal{C}_y$. Note that $\sum_{x,y} \tilde{h}_{xy} \neq V_I$ because some $w_{i,p;j,q}$ might be double counted. The conditions $\mathcal{B}^{(i)}_{(p+1)R} \cap \mathcal{C}_x \neq \varnothing$ and $\mathcal{B}^{(j)}_{(q+1)R} \cap C_y \neq \varnothing$ ensure that we account for all terms $w_{i,p;j,q}(t)$ whose support might overlap with the cubes $\mathcal{C}_x, \mathcal{C}_y$ (Fig. S1). These conditions, together with $\mathrm{dist}(i,j) \leq \ell_{k+1}$, can be relaxed to

1. $\mathrm{dist}(i,x) \leq (p+1)R + R\frac{\sqrt{d}}{2}$,
2. $\mathrm{dist}(j,y) \leq (q+1)R + R\frac{\sqrt{d}}{2}$, and
3. $\mathrm{dist}(x,y) \leq (p+1)R + R\frac{\sqrt{d}}{2} + \ell_{k+1} + (q+1)R + R\frac{\sqrt{d}}{2}$,

where $(p+1)R$ and $(q+1)R$ are the radii of the balls around i and j, $R\sqrt{d}/2$ is the maximum distance between the center and the corner of a hypercube, and the middle term ℓ_{k+1} comes from the maximum distance between i and j.

We bound the norm of $\tilde{h}_{xy}(t)$ using the triangle inequality and relax the conditions for i,j,p,q as we discussed above:

$$\left\|\tilde{h}_{xy}(t)\right\| \leq \sum_{\substack{p,q,i,j\\(1),(2),(3)}} \|w_{i,p;j,q}(t)\| \leq \sum_{\substack{p,q,i,j\\(1),(2),(3)}} \frac{(1+e^{1+\varepsilon})^2}{\mathrm{dist}(i,j)^\alpha} e^{-(p+q)(1+\varepsilon)}, \tag{S20}$$

where the subscript $(1),(2),(3)$ of the sum refers to the three conditions above, respectively. Since $\mathrm{dist}(i,j) \geq \ell_k$, we can simplify the bound and carry out the sums over i,j:

$$\left\|\tilde{h}_{ij}(t)\right\| \leq \frac{(1+e^{1+\varepsilon})^2}{\ell_k^\alpha} \sum_{\substack{p,q,i,j\\(1),(2),(3)}} e^{-(p+q)(1+\varepsilon)} \tag{S21}$$

$$\leq \frac{(1+e^{1+\varepsilon})^2}{\ell_k^\alpha} \sum_{\substack{p,q\\(3)}} 4^d \left(R + pR + R\frac{\sqrt{d}}{2}\right)^d \left(R + qR + R\frac{\sqrt{d}}{2}\right)^d e^{-(p+q)(1+\varepsilon)} \tag{S22}$$

$$= \frac{(1+e^{1+\varepsilon})^2}{\ell_k^\alpha} (2R)^{2d} \sum_{\substack{p,q\\(3)}} \left(p + 1 + \frac{\sqrt{d}}{2}\right)^d \left(q + 1 + \frac{\sqrt{d}}{2}\right)^d e^{-(p+q)(1+\varepsilon)}. \tag{S23}$$

We then use the following identity to simplify the expression: For every $\varepsilon > 0$,

$$x^d \leq g_\varepsilon e^{\varepsilon x} \tag{S24}$$

holds for all $x \geq 0$, where $g_\varepsilon = d!/\varepsilon^d$. Therefore, we can bound

$$\left(p + 1 + \frac{\sqrt{d}}{2}\right)^d \leq g_\varepsilon e^{\varepsilon + \varepsilon\frac{\sqrt{d}}{2}} e^{\varepsilon p}. \tag{S25}$$

Substituting back to the earlier equation, we have

$$\left\|\tilde{h}_{xy}(t)\right\| \leq g_\varepsilon^2 e^{2\varepsilon+2\varepsilon\frac{\sqrt{d}}{2}} \frac{(1+e^{1+\varepsilon})^2}{\ell_k^\alpha}(2R)^{2d} \sum_{\substack{p,q\\(3)}} e^{-(p+q)(1+\varepsilon-\varepsilon)} \leq \frac{\tilde{g}_\varepsilon}{\ell_k^\alpha} R^{2d} \sum_{\substack{p,q\\(3)}} e^{-(p+q)}, \tag{S26}$$

where $\tilde{g}_\varepsilon$ absorbs all constants that depend only on ε and d. Recall that condition (3) is equivalent to

$$p+q \geq \frac{\mathrm{dist}(x,y)}{R} - 2 - \sqrt{d} - \frac{\ell_{k+1}}{R} \equiv a. \tag{S27}$$

We consider two cases. For $q \geq a$, the sum over p can be taken from 0 to ∞:

$$\begin{aligned}\frac{\tilde{g}_\varepsilon}{\ell_k^\alpha} R^{2d} \sum_{q\geq a}\sum_{p\geq 0} e^{-(p+q)} &\leq \frac{\tilde{g}_\varepsilon}{\ell_k^\alpha} R^{2d} e^{-a+1} \sum_{q\geq 0}\sum_{p\geq 0} e^{-(p+q)} = \frac{e^3 \tilde{g}_\varepsilon}{(e-1)^2 \ell_k^\alpha} R^{2d} e^{-a} \\ &= \frac{e^3 \tilde{g}_\varepsilon}{(e-1)^2} e^{2+\sqrt{d}} e^{\frac{\ell_{k+1}}{R}} \frac{R^{2d}}{\ell_k^\alpha} e^{-\frac{\mathrm{dist}(i,j)}{R}}.\end{aligned} \tag{S28}$$

For $q < a$, we sum over $p \geq a - q$:

$$\frac{\tilde{g}_\varepsilon}{\ell_k^\alpha} R^{2d} \sum_{q<a}\sum_{p\geq a-q} e^{-(p+q)} \leq \frac{e^2}{e-1}\frac{\tilde{g}_\varepsilon}{\ell_k^\alpha} R^{2d} g_\varepsilon e^{-(1-\varepsilon)a} \leq \frac{e^2 \tilde{g}_\varepsilon g_\varepsilon e^{2+\sqrt{d}}}{e-1} e^{\frac{\ell_{k+1}}{R}} \frac{R^{2d}}{\ell_k^\alpha} e^{-(1-\varepsilon)\frac{\mathrm{dist}(i,j)}{R}}, \tag{S29}$$

where we have used the identity Eq. (S24) again with $d \geq 1$ and $\varepsilon > 0$ having the same value as before.

Combining Eqs. (S28) and (S29), we have

$$\left\|\tilde{h}_{xy}(t)\right\| \leq \underbrace{\tilde{g}_\varepsilon \frac{e^2}{e-1} e^{2+\sqrt{d}} \left(\frac{e}{e-1} + g_\varepsilon\right) e^{\frac{\ell_{k+1}}{R}} \frac{R^{2d}}{\ell_k^\alpha}}_{=\mathcal{E}_0} e^{-(1-\varepsilon)\frac{\mathrm{dist}(x,y)}{R}}. \tag{S30}$$

Note that $\frac{\mathrm{dist}(x,y)}{R}$ is the rescaled distance between the hypercubes $\mathcal{C}_x$ and $\mathcal{C}_y$. Therefore, the interaction between the hypercubes decays exponentially with the rescaled distance between them. Using the standard Lieb-Robinson bound for exponentially decaying interactions, there exists a constant ν such that

$$\left\|\mathbb{P}_r e^{\mathcal{L}_I t} |O)\right\| \leq \exp\left(\nu \mathcal{E}_0 t - \frac{(1-\varepsilon)r}{R}\right) \tag{S31}$$

for any unit-norm operator O supported on a single hypercube (including operators supported on single sites.) We now choose $R = (1-\varepsilon)\ell_{k+1}$ and rewrite

$$\mathcal{E}_0 = \underbrace{\tilde{g}_\varepsilon \frac{e^2}{e-1} e^{2+\sqrt{d}} \left(\frac{e}{e-1} + g_\varepsilon\right) e^{\frac{1}{1-\varepsilon}} (1-\varepsilon)^{2d}}_{\equiv\lambda} \frac{\ell_{k+1}^{2d}}{\ell_k^\alpha}, \tag{S32}$$

where the constant λ depends only on ε and d. Plugging this expression into the earlier bound, we get

$$\left\|\mathbb{P}_r e^{\mathcal{L}_I t} |O)\right\| \leq \exp\left(\frac{\nu\lambda \frac{\ell_{k+1}^{2d+1}}{\ell_k^\alpha} t - r}{\ell_{k+1}}\right) = \exp\left(\frac{\Delta v\, t - r}{\ell_{k+1}}\right), \tag{S33}$$

where

$$\Delta v \equiv \nu\lambda \frac{\ell_{k+1}^{2d+1}}{\ell_k^\alpha}. \tag{S34}$$

Note that we assume $R = (1-\varepsilon)\ell_{k+1} \geq (1+\varepsilon)\ell_k$. A constant ε satisfying this condition exists as long as $\ell_{k+1} > \ell_k$.

Next, we use the following lemma to "merge" this bound for $e^{\mathcal{L}_I t}$ with the bound in Eq. (S5) for $e^{\mathcal{L}_k t}$.

Lemma S3. Let H_1, H_2 be two possibly time-dependent Hamiltonians and $\mathcal{L}_1, \mathcal{L}_2$ be the corresponding Liouvillians. Suppose that for all unit-norm, single-site operators O and for all times $t \leq \Delta t$ for some Δt,

$$\left\|\mathbb{P}_r e^{\mathcal{L}_1 t} |O)\right\| \leq c_1 r^{\xi_1} e^{\frac{v_1 t - r}{\ell_1}}, \tag{S35}$$

$$\left\|\mathbb{P}_r e^{\mathcal{L}_2 t} |O)\right\| \leq c_2 r^{\xi_2} e^{\frac{v_2 t - r}{\ell_2}}, \tag{S36}$$

for some $\ell_2 \geq \ell_1$ and $c_1, c_2 \geq 1; \xi_1, \xi_2 \geq 0$ are constants. We have

$$\left\|\mathbb{P}_r e^{\mathcal{L}_2 t} e^{\mathcal{L}_1 t} |O)\right\| \leq 2^{d+5} c_1 c_2 r^{\xi_1 + \xi_2 + d + 1} e^{\frac{(v_1+v_2)t - r}{\ell_2}}, \tag{S37}$$

for all $t \leq \Delta t$.

We prove Lemma S3 in Sec. S1 A 1. Using the lemma, we obtain a bound for the evolution under H_{k+1}:

$$\left\|\mathbb{P}_r e^{\mathcal{L}_{k+1} t} |O)\right\| = \left\|\mathbb{P}_r e^{\mathcal{L}_I t} e^{\mathcal{L}_k t} |O)\right\| \leq 2^{d+5} r^{d+1} e^{\frac{(v_k + \Delta v)t - r}{\ell_{k+1}}}. \tag{S38}$$

However, because we assume $v_k t \leq R$ in deriving Eq. (S33), Eq. (S38) is only valid for small time $t \leq (1-\varepsilon)\ell_{k+1}/v_k \equiv \Delta t$. To extend the bound to all time, we use a corollary of Lemma S3:

Corollary 1. Suppose we have a single-site, unit-norm operator O, a Hamiltonian H with a corresponding Liouvillian $\mathcal{L}$, a constant Δt, and

$$\left\|\mathbb{P}_r e^{\mathcal{L} t} |O)\right\| \leq c_0 r^{\xi_0} e^{(vt - r)/\ell} \tag{S39}$$

holds for all $t \leq \Delta t$. Then, for all $t \leq 2^k \Delta t$ for any $k \in \mathbb{N}$, we have

$$\left\|\mathbb{P}_r e^{\mathcal{L} t} |O)\right\| \leq c_k r^{\xi_k} e^{(vt - r)/\ell}, \tag{S40}$$

where $c_k = 2^{(d+5)(2^k - 1)} c_0^{2^k}$, $\xi_k = (2^k - 1)(d+1) + 2^k \xi_0$ are constants. In particular,

$$\left\|\mathbb{P}_r e^{\mathcal{L} t} |O)\right\| \leq e^{\frac{\chi t}{\Delta t} + \frac{vt - r}{\ell}}, \tag{S41}$$

where $\chi = 2[\log(2^{d+5} c_0) + (d + 1 + \xi_0)\log r]$, holds for all time t.

We prove the corollary in Sec. S1 A 2. Using the corollary, we can extend Eq. (S38) to a bound for all time:

$$\left\|\mathbb{P}_r e^{\mathcal{L}_{k+1} t} |O)\right\| \leq \exp\left(\frac{\chi t}{\Delta t} + \frac{(v_k + \Delta v)t - r}{\ell_{k+1}}\right) \leq \exp\left(\frac{\chi_* v_k t}{(1-\varepsilon)\ell_{k+1}} + \frac{(v_k + \Delta v)t - r}{\ell_{k+1}}\right) = \exp\left(\frac{v_{k+1} t - r}{\ell_{k+1}}\right), \tag{S42}$$

where we have upper bounded χ by $\chi_* = 4(d+5)\log 2 + 4(d+1)\log r_*$, $r_* \geq r$ is the diameter of the lattice, and

$$v_{k+1} = \left(\frac{\chi_*}{1-\varepsilon} + 1\right) v_k + \nu\lambda \frac{\ell_{k+1}^{2d+1}}{\ell_k^\alpha} \leq 4(4d+13)\log(r_*) v_k + \nu\lambda \frac{\ell_{k+1}^{2d+1}}{\ell_k^\alpha}. \tag{S43}$$

Here, we have assumed that $r_* \geq 2$ and $\varepsilon \leq 1/2$ so that $1/(1-\varepsilon) \leq 2$, $\chi_* \leq 4(2d+6)\log r_*$, and $1 \leq 4\log r_*$. Therefore, Lemma S1 holds with $\xi = 4(4d+13)$. $\square$

1. *Proof of Lemma S3*

In this section, we prove Lemma S3.

Proof. The bound is trivial for $r < vt$, where $v = v_1 + v_2$. Therefore, we will consider $r \geq vt$ in the rest of the proof. The strategy is to apply Eqs. (S35) and (S36) consecutively. A technical difficulty comes from the fact that after the first evolution $e^{\mathcal{L}_1 t}$, the operator has spread to more than one site. Therefore, we cannot directly apply Eq. (S36), which assumes that the operator is single-site. Instead, we need to use [S4, Lemma 4] to extend the bound for sing-site operators to multi-site operators. In particular, given the assumed bound Eq. (S36) and an unit-norm operator O_X supported on a ball X of radius $x \leq r$, we have

$$\left\|\mathbb{P}_r e^{\mathcal{L}_2 t} |O_X)\right\| \leq \frac{9}{2} |X| c_2 r^{\xi_2} e^{(v_2 t - r + x)/\ell_2}. \tag{S44}$$

With that in mind, we divide the lattice into:

1. A ball of radius $v_1 t$ around the origin,
2. Shells of inner radius $v_1 t + (q-1)\ell_1$ and outer radius $v_1 t + q\ell_1$ for $q = 1, \dots, \frac{r - v_1 t}{\ell_1}$,
3. The rest of the lattice, i.e. sites at least a distance r from the origin.

We then project $e^{\mathcal{L}_1 t} |O)$ into these regions:

$$e^{\mathcal{L}_1 t} |O) = \left[(\mathbb{I} - \mathbb{P}_{v_1 t}) + \sum_{q=1}^{(r-v_1 t)/\ell_1} (\mathbb{P}_{v_1 t + (q-1)\ell_1} - \mathbb{P}_{v_1 t + q\ell_1}) + \mathbb{P}_r \right] e^{\mathcal{L}_1 t} |O) \tag{S45}$$

$$\equiv |O_0) + \sum_{q=1}^{q_*} |O_q) + |O_*) \,, \tag{S46}$$

where $q_* = (r - v_1 t)/\ell_1$. We then apply the other evolution, i.e. $e^{\mathcal{L}_2 t}$, on each term of the above expansion.

First, we consider $|O_0)$, which has norm at most three and is supported on at most $(2v_1 t)^d = (2v_1 t)^d \leq (2r)^d$ sites that are at least a distance $r - v_1 t$ from the outside. Using the assumed bound, we have

$$\left\| \mathbb{P}_r e^{\mathcal{L}_2 t} |O_0) \right\| \leq \frac{27}{2} (2r)^d c_2 r^{\xi_2} e^{(v_2 t - r + v_1 t)/\ell_2} = \frac{27}{2} 2^d c_2 r^{\xi_2 + d} e^{(vt - r)/\ell_2}. \tag{S47}$$

Next, we consider $|O_*)$. Because $\|O_*\| \leq c_1 r^{\xi_1} e^{(v_1 t - r)/\ell_1}$,

$$\left\| \mathbb{P}_r e^{\mathcal{L}_2 t} |O_*) \right\| \leq 2 \|O_*\| \leq 2 c_1 r^{\xi_1} e^{(v_1 t - r)/\ell_1} \leq 2 c_1 c_2 r^{\xi_1 + \xi_2} e^{(vt - r)/\ell_1} \leq 2 c_1 c_2 r^{\xi_1 + \xi_2} e^{(vt - r)/\ell_2}. \tag{S48}$$

Finally, we consider $|O_q)$. Note that O_q is supported on a ball of volume at most $2^d (v_1 t + q\ell_1)^d \leq (2r)^d$, $\|O_q\| \leq (1+e) c_1 r^{\xi_1} e^{-q}$ and the distance between O_q and $\mathbb{P}_r$ is $r - v_1 t - q\ell_1 \leq r$. Therefore, we have

$$\sum_q \left\| \mathbb{P}_r e^{\mathcal{L}_2 t} |O_q) \right\| \leq \sum_q \frac{9}{2} (2r)^d \|O_q\| \, c_2 r^{\xi_2} e^{(v_2 t - (r - v_1 t - q\ell_1))/\ell_2} \tag{S49}$$

$$\leq \sum_q \frac{9}{2} (2r)^d (1+e) c_1 r^{\xi_1} e^{-q} c_2 r^{\xi_2} e^{(vt - r)/\ell_2} e^{q \frac{\ell_1}{\ell_2}} \tag{S50}$$

$$\leq \sum_q 17 \times 2^d c_1 c_2 r^{\xi_1 + \xi_2 + d} e^{(vt - r)/\ell_2} \tag{S51}$$

$$\leq 17 \times 2^d c_1 c_2 r^{\xi_1 + \xi_2 + d + 1} e^{(vt - r)/\ell_2}, \tag{S52}$$

where we have used $\ell_1 \leq \ell_2$ and the fact that there are at most $\frac{r - v_1 t}{\ell_1} \leq r$ different q.

Combining Eqs. (S47), (S48) and (S52) with $c_1, c_2 \geq 1$, $d \geq 1$ and $\xi_1, \xi_2 \geq 0$, we have

$$\left\| \mathbb{P}_r e^{\mathcal{L}_2 t} e^{\mathcal{L}_1 t} |O) \right\| \leq 2^{d+5} c_1 c_2 r^{\xi_1 + \xi_2 + d + 1} e^{(vt - r)/\ell_2}, \tag{S53}$$

with $v = v_1 + v_2$. Therefore, the lemma follows. ☐

2. *Proof of Corollary 1*

In this section, we prove Corollary 1, an application of Lemma S3 which extends the validity of a bound from $t \leq \Delta t$ to arbitrary time.

Proof. The lemma clearly holds for $k = 0$. So we will prove it by induction. Suppose Eq. (S40) holds for some $k \in \mathbb{N}$. We will prove that it holds for $k + 1$.

The strategy is to apply the assumed bound for k [Eq. (S40)] twice:

$$\left\| \mathbb{P}_r e^{\mathcal{L} t} |O) \right\| = \left\| \mathbb{P}_r e^{\mathcal{L} t/2} e^{\mathcal{L} t/2} |O) \right\|, \tag{S54}$$

where the evolutions under $e^{\mathcal{L} t/2}$ can be bounded by the assumed bound because $t/2 \leq 2^k \Delta t$. We then use Lemma S3 to merge the two identical bounds with $v_1 = v_2 \to v/2$, $\ell_1 = \ell_2 \to \ell$, $c_1 = c_2 \to c_k$, $\xi_1 = \xi_2 \to \xi_k$:

$$\left\| \mathbb{P}_r e^{\mathcal{L} t} |O) \right\| \leq 2^{d+5} c_k^2 r^{2\xi_k + d + 1} e^{(vt - r)/\ell}. \tag{S55}$$

We choose

$$c_{k+1} = 2^{d+5} c_k^2 \qquad \Rightarrow c_k = 2^{(d+5)(2^k-1)} c_0^{2^k} \tag{S56}$$

$$\xi_{k+1} = 2\xi_k + d + 1 \qquad \Rightarrow \xi_k = (2^k - 1)(d+1) + 2^k \xi_0. \tag{S57}$$

Therefore, by induction, Eq. (S40) holds for $k+1$.

Next, to prove Eq. (S41), we choose $k = \lceil \log_2(t/\Delta t) \rceil$ so that $t \le 2^k \Delta t$. We also have $2^k \le \frac{2t}{\Delta t}$. Therefore, $c_k \le (2^{d+5} c_0)^{\frac{2t}{\Delta t}}$, $\xi_k \le (d+1+\xi_0)\frac{2t}{\Delta t}$. Plugging them into Eq. (S40), we have

$$\left\| \mathbb{P}_r e^{\mathbb{P}_r t} |O) \right\| \le (2^{d+5} c_0)^{\frac{2t}{\Delta t}} r^{(d+1+\xi_0)\frac{2t}{\Delta t}} e^{\frac{vt-r}{\ell}} = e^{\chi \frac{t}{\Delta t} + \frac{vt-r}{\ell}}, \tag{S58}$$

with $\chi = 2[\log(2^{d+5} c_0) + (d+1+\xi_0)\log r]$. □

B. Proof of Lemma S2

In this section, we prove Lemma S2.

Proof. First, we need the following lemma, which uses an existing bound to prove a tighter bound. We will use the lemma recursively to prove the nearly optimal bound in Lemma S2.

Lemma S4. Let $\eta \in (0, \frac{1}{\alpha - d})$ be an arbitrary constant and

$$\delta = \frac{2d+1}{(2d+1-\alpha)(1+\eta(2d+1-\alpha))} \tag{S59}$$

be another constant. Suppose there exist constants $\gamma, C, c \ge 0$, $\kappa \ge \delta$, and $\beta > d$ such that

$$\left\| \mathbb{P}_r e^{\mathcal{L}t} |O) \right\| \le C \log^{\kappa} r_* \frac{t^{\gamma}}{r^{\beta}}, \tag{S60}$$

holds for all $t^{\gamma} \le c r^{\beta} / \log^{\delta} r_*$. Then, there exist constants $C', c' > 0$, and $\kappa' > \delta$ such that

$$\left\| \mathbb{P}_r e^{\mathcal{L}t} |O) \right\| \le C' \log^{\kappa'} r_* \frac{t^{\gamma'}}{r^{\beta'}}, \tag{S61}$$

holds for all $t^{\gamma'} \le c' r^{\beta'} / \log^{\delta} r_*$, where

$$\kappa' = \max\left\{ \kappa - \frac{\delta(\beta - d)}{\beta} + \frac{\alpha - d}{2d+1-\alpha}, \delta \right\}, \tag{S62}$$

$$\gamma' = \gamma d/\beta + 1 - \eta(\alpha - d), \tag{S63}$$

$$\beta' = \alpha - d - \eta(\alpha - 2d)(\alpha - d) > d. \tag{S64}$$

Proof. Let $V = H - H_n$ to be the sum over interactions of range more than ℓ_n. We have [S3]

$$e^{\mathcal{L}t} |O) = e^{\mathcal{L}_n t} |O) + \sum_{h_{ij}} \int_0^t ds\; e^{\mathcal{L}(t-s)} \mathcal{L}_{h_{ij}} e^{\mathcal{L}_n s} |O), \tag{S65}$$

where the sum is over all h_{ij} in V. The first term is the evolution under H_n, which we can bound using Eq. (S8). Our task is to bound the second term.

Without loss of generality, we assume $i \le j$. Because $\mathcal{L}_{h_{ij}} |O)$ only acts nontrivially on the part of O supported at least a distance $\mathrm{dist}(i,0)$ from the origin, we can insert $\mathbb{P}_{\mathrm{dist}(i,0)}$ in the middle of the intergrand and use the triangle inequality:

$$\left\| \mathbb{P}_r e^{\mathcal{L}t} |O) \right\| \le \left\| \mathbb{P}_r e^{\mathcal{L}_n t} |O) \right\| + \left\| \mathbb{P}_r \sum_{h_{ij}} \int_0^t ds e^{\mathcal{L}(t-s)} \mathcal{L}_{h_{ij}} \mathbb{P}_{\mathrm{dist}(i,0)} e^{\mathcal{L}_n s} |O) \right\| \tag{S66}$$

$$\le \left\| \mathbb{P}_r e^{\mathcal{L}_n t} |O) \right\| + 4 \sum_{h_{ij}} \int_0^t ds \left\| h_{ij} \right\| \left\| \mathbb{P}_{\mathrm{dist}(i,0)} e^{\mathcal{L}_n s} |O) \right\|. \tag{S67}$$

Because $\|h_{ij}\| \leq 1/\text{dist}(i,j)^\alpha$ and $\text{dist}(i,j) > \ell_n$, there exist a constant K_1 such that $\sum_{j:\text{dist}(i,j)>\ell_n} \|h_{ij}\| \leq K_1/\ell_n^{\alpha-d}$ for all $i \in \Lambda$. Therefore, we have

$$\left\|\mathbb{P}_r e^{\mathcal{L}t} |O)\right\| \leq \left\|\mathbb{P}_r e^{\mathcal{L}_n t} |O)\right\| + \frac{4K_1}{\ell_n^{\alpha-d}} \int_0^t ds \sum_i \left\|\mathbb{P}_{\text{dist}(i,0)} e^{\mathcal{L}_n s} |O)\right\|, \tag{S68}$$

We then consider two cases for the sum over i. If $\text{dist}(i,0)^\beta \leq \frac{1}{c}\log^\delta(r_*)\ s^\gamma$, we use a trivial bound on the projection:

$$\sum_{i:\text{dist}(i,0)^\beta \leq \frac{1}{c}\log^\delta(r_*)\ s^\gamma} \left\|\mathbb{P}_{\text{dist}(i,0)} e^{\mathcal{L}_n s} |O)\right\| \leq (2c^{-1/\beta}(\log r_*)^{\delta/\beta} s^{\gamma/\beta})^d \times 2 = 2^{d+1} c^{-d/\beta} (\log r_*)^{\delta d/\beta} s^{\gamma d/\beta}. \tag{S69}$$

Otherwise, if $\text{dist}(i,0)^\beta > \frac{1}{c}\log^\delta(r_*)\ s^\gamma$, we apply Eq. (S60):

$$\begin{aligned}
\sum_{i:\text{dist}(i,0)^\beta > \frac{1}{c}\log^\delta(r_*)\ s^\gamma} \left\|\mathbb{P}_{\text{dist}(i,0)} e^{\mathcal{L}_n s} |O)\right\| &\leq C \log^\kappa r_* \sum_{i:\text{dist}(i,0)^\beta > \frac{1}{c}\log^\delta(r_*)\ s^\gamma} \frac{s^\gamma}{\text{dist}(i,0)^\beta} && \text{(S70)}\\
&\leq C K_2 \log^\kappa r_* \frac{s^\gamma}{[c^{-1/\beta}(\log r_*)^{\delta/\beta} s^{\gamma/\beta}]^{\beta-d}} && \text{(S71)}\\
&\leq C K_2 c^{\frac{\beta-d}{\beta}} (\log r_*)^{\kappa - \frac{\delta(\beta-d)}{\beta}}\ s^{\gamma d/\beta}, && \text{(S72)}
\end{aligned}$$

where K_2 is a constant such that

$$\sum_{i:\text{dist}(i,0)>a} \frac{1}{\text{dist}(i,0)^\beta} \leq \frac{K_2}{a^{\beta-d}}, \tag{S73}$$

for all $a > 0$. Such a constant K_2 exists because $\beta > d$ by assumption.

Combining Eqs. (S69) and (S72) and accounting for $\kappa \geq \delta$, we can upper bound

$$\frac{4K_1}{\ell_n^{\alpha-d}} \int_0^t ds \sum_i \left\|\mathbb{P}_{\text{dist}(i,0)} e^{\mathcal{L}_n s} |O)\right\| \leq K (\log r_*)^{\kappa - \frac{\delta(\beta-d)}{\beta}} \frac{t^{\frac{\gamma d}{\beta}+1}}{\ell_n^{\alpha-d}}, \tag{S74}$$

where we absorb all constants into $K = 4K_1(2^{d+1}c^{-d/\beta} + CK_2/c^{\frac{\beta-d}{\beta}})\frac{\beta}{\gamma d+\beta}$. Substituting Eq. (S74) in Eq. (S68), we have a bound for the evolution under H:

$$\left\|\mathbb{P}_r e^{\mathcal{L}t} |O)\right\| \leq e^{\frac{v_n t - r}{\ell_n}} + K(\log r_*)^{\kappa - \frac{\delta(\beta-d)}{\beta}} \frac{t^{\frac{\gamma d}{\beta}+1}}{\ell_n^{\alpha-d}} \tag{S75}$$

We now substitute the values of v_n and ℓ_n into the bound. Recall from Eq. (S3) that

$$n = \left\lfloor \frac{1}{\log L} \log\left[r\left(\frac{t}{r^{\alpha-2d}}\right)^\eta\right]\right\rfloor, \tag{S76}$$

where $\eta \in (0, \frac{1}{\alpha-d})$ is an arbitrary small constant. With this choice, we can bound ℓ_n from both above and below:

$$r \geq r\left(\frac{t}{r^{\alpha-2d}}\right)^\eta \geq \ell_n = L^n \geq \frac{r}{L}\left(\frac{t}{r^{\alpha-2d}}\right)^\eta = \frac{r}{(\xi \log r_*)^{1/(2d+1-\alpha)}}\left(\frac{t}{r^{\alpha-2d}}\right)^\eta. \tag{S77}$$

With $v_1 = 4e\tau\ell_1$, $x = \xi \log r_* = L^{2d+1-\alpha}$, we also have a bound for v_n from Eq. (S10):

$$v_n \leq r^{2d+1-\alpha}\left(\frac{t}{r^{\alpha-2d}}\right)^{\eta(2d+1-\alpha)} \frac{4e\tau x^{1/(2d+1-\alpha)} + \nu\lambda(n-1)x^{(2d+1)/(2d+1-\alpha)}}{x}. \tag{S78}$$

Assuming that $r_* \geq e^{e^{2d+1-\alpha}/\xi}$ so that $\log L \geq 1$ and $x \geq 1$, we have $n \leq \log r/\log L \leq \log r_* = x/\xi$. We can then crudely upper bound

$$v_n \leq (4e\tau + \nu\lambda/\xi)(\xi \log r_*)^{\frac{2d+1}{2d+1-\alpha}}\ \frac{r}{t}\left(\frac{t}{r^{\alpha-2d}}\right)^{1+\eta(2d+1-\alpha)} = K_3(\log r_*)^{\frac{2d+1}{2d+1-\alpha}} \frac{r}{t}\left(\frac{t}{r^{\alpha-2d}}\right)^{1+\eta(2d+1-\alpha)}. \tag{S79}$$

where K_3 is a constant. Assuming

$$t \leq r^{\alpha-2d}/[2K_3(\log r_*)^{\frac{2d+1}{2d+1-\alpha}}]^{\frac{1}{1+\eta(2d+1-\alpha)}} \tag{S80}$$

so that $v_n t \leq r/2$, we can simplify the first term of Eq. (S75):

$$e^{\frac{v_n t - r}{\ell_n}} \leq e^{-\frac{r}{2\ell_n}} \leq \exp\left[-\frac{1}{2}\left(\frac{r^{\alpha-2d}}{t}\right)^{\eta}\right]. \tag{S81}$$

Similarly, the second term of Eq. (S75) can be simplified to

$$\begin{aligned} K(\log r_*)^{\kappa-\frac{\delta(\beta-d)}{\beta}}\frac{t^{\gamma d/\beta+1}}{\ell_n^{\alpha-d}} &\leq K(\log r_*)^{\kappa-\frac{\delta(\beta-d)}{\beta}}t^{\gamma d/\beta+1}\left[\frac{r}{(\xi\log r_*)^{1/(2d+1-\alpha)}}\left(\frac{t}{r^{\alpha-2d}}\right)^{\eta}\right]^{d-\alpha} && \text{(S82)}\\ &= K\xi^{(\alpha-d)/(2d+1-\alpha)}(\log r_*)^{\kappa-\frac{\delta(\beta-d)}{\beta}+\frac{\alpha-d}{2d+1-\alpha}}\frac{t^{\gamma d/\beta+1-\eta(\alpha-d)}}{r^{\alpha-d-\eta(\alpha-2d)(\alpha-d)}} && \text{(S83)}\\ &= K_4\log^{\kappa'} r_*\frac{t^{\gamma'}}{r^{\beta'}}, && \text{(S84)} \end{aligned}$$

where K_4, κ', γ' are constants. In particular, $\gamma' = \gamma d/\beta + 1 - \eta(\alpha-d)$ and $\beta' = \alpha - d - \eta(\alpha-2d)(\alpha-d) > d$. Combining Eqs. (S81) and (S84), we get a bound for the evolution under H.

We now simplify the bound by considering t such that

$$t^{\gamma'} \leq \frac{c' r^{\beta'}}{(\log r_*)^{\delta\gamma'}} \tag{S85}$$

for some constant c' and $\delta = \frac{2d+1}{(2d+1-\alpha)(1+\eta(2d+1-\alpha))}$. Since $\beta/\gamma \leq \alpha - 2d$ by assumption, we also have

$$\frac{\beta'}{\gamma'} = \frac{\alpha-d-\eta(\alpha-2d)(\alpha-d)}{\frac{\gamma d}{\beta}+1-\eta(\alpha-d)} \leq \frac{\alpha-d-\eta(\alpha-2d)(\alpha-d)}{\frac{d}{\alpha-2d}+1-\eta(\alpha-d)} = \alpha-2d. \tag{S86}$$

Therefore, with $c' = (2K_3)^{\frac{-\gamma'}{(1+\eta(2d+1-\alpha))}}$, Eq. (S85) satisfies the condition in Eq. (S80). In addition, for $r^{\alpha-2d} \geq t$, there exists a constant K_5 such that

$$\exp\left[-\frac{1}{2}\left(\frac{r^{\alpha-2d}}{t}\right)^{\eta}\right] \leq K_5\left(\frac{t}{r^{\alpha-2d}}\right)^{\gamma'} \leq K_5\frac{t^{\gamma'}}{r^{\beta'}}, \tag{S87}$$

where we have again used $\gamma'(\alpha-2d) \geq \beta'$ in the last inequality. Replacing Eq. (S81) by Eq. (S87) and combining with Eq. (S84), we arrive at a bound

$$\left\|\mathbb{P}_r e^{\mathcal{L}t}|O)\right\| \leq C'\log^{\kappa'} r_*\frac{t^{\gamma'}}{r^{\beta'}}, \tag{S88}$$

for all $t^{\gamma'} \leq c' r^{\beta'}/\log^{\delta} r_*$, where $C' \geq K_4 + K_5$ and c' are constants,

$$\begin{aligned} \kappa' &= \kappa - \frac{\delta(\beta-d)}{\beta} + \frac{\alpha-d}{2d+1-\alpha}, && \text{(S89)}\\ \gamma' &= \gamma d/\beta + 1 - \eta(\alpha-d), && \text{(S90)}\\ \beta' &= \alpha - d - \eta(\alpha-2d)(\alpha-d) > d. && \text{(S91)} \end{aligned}$$

If $\kappa' < \delta$, we simply replace κ' by δ in Eq. (S88). Such replacement can only increase the bound in Eq. (S88). Therefore, Lemma S4 follows. □

We now use Lemma S4 to prove Lemma S2. To satisfy the assumption of Lemma S4, we start with the bound in Ref. [S5]: There exist constants K_6, K_7, and v_F such that

$$\left\|[O', e^{\mathcal{L}t}O]\right\| \leq K_6\exp\left(v_F t - \frac{r}{t^{(1+d)/(\alpha-2d)}}\right) + K_7\frac{t^{\frac{\alpha(\alpha-d+1)}{\alpha-2d}}}{r^{\alpha}}, \tag{S92}$$

for all single-site, unit-norm operators O' supported a distance r from O. We consider the regime

$$t^{\frac{\alpha(\alpha-d+1)}{\alpha-2d}} \leq cr^{\alpha-d}/\log^{\delta} r_* \leq cr^{\alpha-d} \leq cr^{\alpha}, \tag{S93}$$

where we choose $c = (2v_F)^{-\alpha}$ so that

$$v_F t \leq \frac{r}{2t^{(1+d)/(\alpha-2d)}}. \tag{S94}$$

Therefore, there exists a constant K_8 such that

$$K_6 \exp\left(v_F t - \frac{r}{t^{(1+d)/(\alpha-2d)}}\right) \leq K_6 \exp\left(-\frac{r}{2t^{(1+d)/(\alpha-2d)}}\right) \leq K_8 \left(\frac{t^{(1+d)/(\alpha-2d)}}{r}\right)^{\frac{\alpha(\alpha-d+1)}{1+d}} \leq K_8 \frac{t^{\alpha(\alpha-d+1)/(\alpha-2d)}}{r^{\alpha}} \tag{S95}$$

holds for all t satisfying Eq. (S93). In the last inequality, we have used $\alpha - d + 1 \geq d + 1$ to lower bound the exponent of r. Substituting Eq. (S95) into Eq. (S92), we get a simplified version of the bound in Ref. [S5]:

$$\left\|\left[O', e^{\mathcal{L}t} O\right]\right\| \leq K_9 \frac{t^{\frac{\alpha(\alpha-d+1)}{\alpha-2d}}}{r^{\alpha}}, \tag{S96}$$

where $K_9 = K_7 + K_8$. Applying Lemma 4 in Ref. [S4], there exists a constant K_{10} such that:

$$\left\|\mathbb{P}_r e^{\mathcal{L}t} \left|O\right)\right\| \leq K_{10} \frac{t^{\frac{\alpha(\alpha-d+1)}{\alpha-2d}}}{r^{\alpha-d}} < K_{10} \log^{\delta} r_* \frac{t^{\frac{\alpha(\alpha-d+1)}{\alpha-2d}}}{r^{\alpha-d}}, \tag{S97}$$

where the additional factor $-d$ in the exponent of r comes from "integrating" over sites that are at least a distance r from the origin. Equation (S97) satisfies the assumption of Lemma S4, with $C \to K_{10}, c \to (2v_F)^{-\alpha}, \kappa \to \delta, \gamma \to \frac{\alpha(\alpha-d+1)}{\alpha-2d}, \beta \to \alpha - d$. Therefore, by the lemma, there exist constants C_1, c_1, κ_1 such that

$$\left\|\mathbb{P}_r e^{\mathcal{L}_r} \left|O\right)\right\| \leq C_1 \log^{\kappa_1} r_* \frac{t^{\gamma_1}}{r^{\beta_1}} \tag{S98}$$

holds for all $t^{\gamma_1} \leq c_1 r^{\beta_1}/\log^{\delta} r_*$, where

$$\gamma_1 = \frac{\alpha(\alpha-d+1)}{\alpha-2d}\frac{d}{\alpha-d} + 1 - \eta(\alpha-d), \tag{S99}$$

$$\beta_1 = \alpha - d - \eta(\alpha-2d)(\alpha-d). \tag{S100}$$

Equation (S98) again satisfies the assumption of Lemma S4. Applying the lemma again with $\gamma \to \gamma_1, \beta \to \beta_1$, we obtain

$$\left\|\mathbb{P}_r e^{\mathcal{L}_r} \left|O\right)\right\| \leq C_2 \log^{\kappa_2} r_* \frac{t^{\gamma_2}}{r^{\beta_2}} \tag{S101}$$

for some constants $C_2, \kappa_2, \beta_2 = \beta_1$, and

$$\gamma_2 = \frac{\gamma_1 d}{\beta_1} + 1 - \eta(\alpha-d) \equiv f(\gamma_1). \tag{S102}$$

After applying Lemma S4 for m times, we obtain

$$\left\|\mathbb{P}_r e^{\mathcal{L}_r} \left|O\right)\right\| \leq C_m \log^{\kappa_m} r_* \frac{t^{\gamma_m}}{r^{\beta_m}} = C_m \log^{\kappa_m} r_* \left(\frac{t}{r^{\beta_m/\gamma_m}}\right)^{\gamma_m} \tag{S103}$$

for some constants $C_m, \kappa_m, \beta_m = \beta_1$, and

$$\gamma_m = f^{\circ(m-1)}(\gamma_1), \tag{S104}$$

where $f^{\circ(m-1)}$ denotes the $(m-1)$-th composition of the function f. It is straightforward to show that

$$\lim_{\eta\to 0}\lim_{m\to\infty} \gamma_m = \lim_{\eta\to 0} \frac{\alpha - d - \eta(\alpha-2d)(\alpha-d)}{\alpha-2d} = \frac{\alpha-d}{\alpha-2d}, \tag{S105}$$

$$\lim_{\eta\to 0}\lim_{m\to\infty} \frac{\beta_m}{\gamma_m} = \alpha - 2d. \tag{S106}$$

Therefore, for all $\varepsilon > 0$, there exist $m \geq 1, \eta \in (0, \frac{1}{\alpha-d})$ such that $\beta_m/\gamma_m \geq \alpha - 2d - \varepsilon$ and $\gamma_m \geq \frac{\alpha-d}{\alpha-2d} - \varepsilon$. We obtain

$$\left\|\mathbb{P}_r e^{\mathcal{L}_r} |O)\right\| \leq C_m \log^{\kappa_m} r_* \left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon}, \tag{S107}$$

which holds for all $t \leq c_m^{1/\gamma_m} r^{\alpha-2d-\varepsilon}/(\log r_*)^{\delta/\gamma_m} \leq r^{\alpha-2d-\varepsilon}$. Lemma S2 thus follows. □

C. Removing the dependence on the lattice size

In this section, we use Lemma S2 to prove Theorem 1 by removing the dependence on r_*. Since the bound in Lemma S2 depends on $\log(r_*)$, this dependence is mild for all r_* that scale polynomially as a function of r. On the other hand, if r_* scales as a super-polynomial function of r, we intuitively expect interactions supported at distance $\propto r_*$ from the origin to play a very minor role in the evolution of O.

Our strategy is to first truncate away interactions supported beyond a distance $r_0 = \text{poly}(r) < r_*$ and apply Lemma S2 to obtain a bound for the truncated lattice. We then use the interaction-picture technique to add these interactions back into the bound and show that their contributions add up to a small r_*-independent constant that we can control using r_0.

Proof. Let $H_{\text{out}} = \mathbb{P}_{r_0} H$ denote the terms of the Hamiltonian H that have support outside a distance r_0 from the origin, $H_{\text{in}} = H - H_{\text{out}}$ be the rest of the Hamiltonian, and $\mathcal{L}_{\text{out}}$, $\mathcal{L}_{\text{in}}$ are the corresponding Liouvillians. Using the triangle inequality, we have

$$\left\|\mathbb{P}_r e^{\mathcal{L}t} |O)\right\| \leq \left\|\mathbb{P}_r e^{\mathcal{L}_{\text{in}}t} |O)\right\| + \sum_{h_{ij}} \left\|\mathbb{P}_r \int_0^t ds\ e^{\mathcal{L}(t-s)} \mathcal{L}_{h_{ij}} e^{\mathcal{L}_{\text{in}}s} |O)\right\|, \tag{S108}$$

where the sum is taken over terms h_{ij} in H_{out}. Without loss of generality, we assume $\text{dist}(i,0) \leq \text{dist}(j,0)$, which implies $\text{dist}(j,0) \geq r_0$. In addition, since $e^{\mathcal{L}_{\text{in}}s} |O)$ is supported entirely within the radius r_0 from the origin, only terms where $\text{dist}(i,0) \leq r_0$ contribute to the above sum. We consider two cases: $\text{dist}(i,0) > r_0/2$ and $\text{dist}(i,0) \leq r_0/2$.

In the former case, we insert $\mathbb{P}_{\text{dist}(i,0)}$ in the middle of the integrand and bound

$$\begin{aligned}
&\sum_{h_{ij}:\text{dist}(i,0)\in(\frac{r_0}{2},r_0]} \left\|\mathbb{P}_r \int_0^t ds\ e^{\mathcal{L}(t-s)} \mathcal{L}_{h_{ij}} \mathbb{P}_{\text{dist}(i,0)} e^{\mathcal{L}_{\text{in}}s} |O)\right\| \leq 4 \sum_{h_{ij}:\text{dist}(i,0)\in(\frac{r_0}{2},r_0]} \|h_{ij}\| \int_0^t ds \left\|\mathbb{P}_{\text{dist}(i,0)} e^{\mathcal{L}_{\text{in}}s} |O)\right\| \\
&\leq 4K_1 C \log^{\kappa}(2r_0) \sum_{i:\text{dist}(i,0)\in(\frac{r_0}{2},r_0]} \int_0^t ds \left(\frac{t}{\text{dist}(i,0)^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon} \leq K_2 r_0^d t \left(\frac{t}{r_0^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon},
\end{aligned} \tag{S109}$$

where K_2 is a constant. We have used Lemma S2 to bound the evolution under $e^{\mathcal{L}_{\text{in}}s}$, which is supported entirely within a truncated lattice of diameter $2r_0$, and used the fact that the interaction h_{ij} decays as a power law with an exponent $\alpha > 2d$ to bound the sum over j by a constant. We require $t \leq c_1 r^{\alpha-2d-\varepsilon}/\log^{\delta}(2r_0)$, for some constant c_1, δ, to satisfy the conditions of Lemma S2.

On the other hand, when $\text{dist}(i,0) \leq r_0/2$, we have $\text{dist}(i,j) \geq r_0/2$. Therefore, there exists a constant c_2 such that $\sum_j \|h_{ij}\| \leq c_2/r_0^{\alpha-d}$ for all i. We can then bound

$$\sum_{h_{ij}:\text{dist}(i,0)\leq\frac{r_0}{2}} \left\|\mathbb{P}_r \int_0^t ds\ e^{\mathcal{L}(t-s)} \mathcal{L}_{h_{ij}} e^{\mathcal{L}_{\text{in}}s} |O)\right\| \leq 4 \sum_{i:\text{dist}(i,0)\leq\frac{r_0}{2}} \frac{c_2}{r_0^{\alpha-d}} \int_0^t ds \leq K_3 \frac{t}{r_0^{\alpha-2d}}, \tag{S110}$$

for some constant K_3.

Using Lemma S2 on the first term of Eq. (S108) and combining with Eqs. (S109) and (S110), we have:

$$\left\|\mathbb{P}_r e^{\mathcal{L}t} |O)\right\| \leq C \log^{\kappa}(2r_0) \left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon} + K_2 r_0^d t \left(\frac{t}{r_0^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon} + K_3 \frac{t}{r_0^{\alpha-2d}}. \tag{S111}$$

We choose $r_0 = r^{\xi}$, where

$$\xi = \left(1 + \frac{\alpha-d}{\alpha-2d} - \varepsilon\right) \frac{\alpha-2d-\varepsilon}{\alpha-2d-\varepsilon\frac{(\alpha-2d)^2+\alpha-d}{\alpha-2d}+\varepsilon^2} \geq \frac{\alpha-d}{\alpha-2d}, \tag{S112}$$

and we require $\varepsilon \le (\alpha-2d)^2/[(\alpha-2d)^2+\alpha-d]$ so that the lower bound on ξ holds. Under this choice,

$$K_2 r_0^d t \left(\frac{t}{r_0^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon} = K_2 \left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{1+\frac{\alpha-d}{\alpha-2d}-\varepsilon} \le K_4 \left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon}, \tag{S113}$$

for all $t \le c_1 r^{\alpha-2d-\varepsilon}$, where K_4 is a constant. In addition, for $\varepsilon \le (\alpha-2d)^2/[(\alpha-2d)^2+\alpha-d]$, $\xi \ge (\alpha-d)/(\alpha-2d)$ and, therefore,

$$K_3 \frac{t}{r_0^{\alpha-2d}} \le K_3 \frac{t}{r^{\alpha-d}}. \tag{S114}$$

Combining Eqs. (S112) to (S114), we have

$$\left\|\mathbb{P}_r e^{\mathcal{L}t} |O)\right\| \le K_5 \log^\kappa(r) \left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\varepsilon} + K_6 \frac{t}{r^{\alpha-d}}, \tag{S115}$$

which holds for all $t \le c_1 r^{\alpha-2d-\varepsilon}/\log^\delta(2r^\xi)$ for some constants K_5, K_6 independent of t, r.

Next, we simplify Eq. (S115) by "hiding" the factor $\log^\kappa r$ inside the constant ε. Specifically, there exist a constant K_7 such that $\log^\kappa r \le K_7 r^{\varepsilon'}$, where $\varepsilon' = \frac{\varepsilon}{2}\left(\frac{\alpha-d}{\alpha-2d} - \frac{\varepsilon}{2}\right)$, and constants K_8, K_9 such that

$$\left\|\mathbb{P}_r e^{\mathcal{L}t} |O)\right\| \le K_8 \log^\kappa(r) \left(\frac{t}{r^{\alpha-2d-\varepsilon/2}}\right)^{\frac{\alpha-d}{\alpha-2d}-\frac{\varepsilon}{2}} + K_9 \frac{t}{r^{\alpha-d}} \le K_7 K_8 \left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\frac{\varepsilon}{2}} + K_9 \frac{t}{r^{\alpha-d}}, \tag{S116}$$

which holds for all $t \le c_1 r^{\alpha-2d-\varepsilon/2}/\log^\delta(2r^\xi)$. In addition, there exists a constant K_{10} such that $K_{10}\log^\delta(2r^\xi) \le r^{\varepsilon/2}$ for all $r \ge 1$. By requiring that $t \le K_{10} c_1 r^{\alpha-2d-\varepsilon}$, we also ensure $t \le c_1 r^{\alpha-2d-\varepsilon/2}/\log^\delta(2r^\xi)$. Therefore, Theorem 1 follows with $c \to c_1 K_{10}, C_1 \to K_7 K_8$, and $C_2 \to K_9$.

□

S2. APPLICATIONS OF THEOREM 1

We discussed in the main text that the tightened light cone and nearly optimal tail in Theorem 1 improved the scaling for various applications of Lieb-Robinson bounds to problems of physical interest in the regime $2d < \alpha < 2d+1$ Here we provide some mathematical details to justify those assertions. We also provide a table briefly summarizing the bounds we will use to compare, where we consider each bound to take the form $\|[A(t), B]\| \le ct^\gamma/r^\beta$ for some constants c, γ and β, where A is single-site, but B may generally be some large multi-site operator.

Bound	Light cone	Tail	γ	β	γ'	β'
This work (B1)	$t \gtrsim r^{\alpha-2d}$	$1/r^{\alpha-d}$	$\frac{\alpha-d}{\alpha-2d}$	$\alpha-d$	$\gamma+1$	β
Ref. [S5] (B2)	$t \gtrsim r^{\frac{(\alpha-d)(\alpha-2d)}{\alpha(\alpha-d+1)}}$	$1/r^{\alpha-d}$	$\frac{\alpha(\alpha-d+1)}{\alpha-2d}$	$\alpha-d$	$\gamma+1$	β
Ref. [S6] (B3)	$t \gtrsim r^{\frac{\alpha-2d}{\alpha-d}}$	$1/r^{\alpha-2d}$	$\alpha-d$	$\alpha-2d$	γ	β

TABLE S1. Comparison of Lieb-Robinson bounds for $2d < \alpha < 2d+1$. We name the bounds B1, B2, and B3 for brevity. We ignore the arbitrarily small parameter ε in B1 for simplicity, as it does not affect the conclusions.

We first consider the application of the bound on the growth of connected correlators. Consider two unit-norm, single-site observables A and B initially supported on sites x and y, respectively, such that x and y are separated by a distance r. Let $|\psi\rangle$ be a product state between $\mathcal{B}_{r/2}(x)$ and $\mathcal{B}_{r/2}(y)$, where $\mathcal{B}_{r/2}(x)$ is the ball of radius $r/2$ around x. The connected correlator is defined by

$$C(r,t) \equiv \langle A(t)B(t)\rangle - \langle A(t)\rangle \langle B(t)\rangle, \tag{S117}$$

where $\langle\cdot\rangle \equiv \langle\psi|\cdot|\psi\rangle$. Define $\tilde{A}(t) \equiv \mathrm{Tr}_{\mathcal{B}^c_{r/2}(x)}[A(t)]$ and $\tilde{B}$ similarly. It is elementary to bound $C(r,t)$ by

$$C(r,t) \le 2\left\|A(t)-\tilde{A}(t)\right\| + 2\left\|B(t)-\tilde{B}(t)\right\|. \tag{S118}$$

That is, the connected correlator is controlled by the error in truncating $A(t)$ and $B(t)$ to within a ball of radius $r/2$ around their initial support. A simple result from Ref. [S7] allows us to bound this error

$$\left\| A(t) - \tilde{A}(t) \right\| \leq \int_{\mathcal{B}^c_{r/2}(x)} dU \, \|[U, A(t)]\| \leq c \frac{t^\gamma}{(r/2)^\beta}, \tag{S119}$$

where dU is the Haar measure on unitaries supported outside a ball of radius $r/2$ around x. Thus, for a given Lieb-Robinson bound

$$C(r,t) \leq 2^{\beta+2} c \frac{t^\gamma}{r^\beta}. \tag{S120}$$

Ignoring constants and focusing on the asymptotics with respect to t and r, we see that

$$R_{12} \equiv \frac{\text{B1}}{\text{B2}} \sim \left(\frac{t^{\alpha-d}}{t^{\alpha(\alpha-d+1)}} \right)^{\frac{1}{\alpha-2d}}, \tag{S121}$$

$$R_{13} \equiv \frac{\text{B1}}{\text{B3}} \sim \left(\frac{t^{\frac{1}{\alpha-2d}}}{t} \right)^{\alpha-d} r^{-d}. \tag{S122}$$

Thus, as t increases, the tighter light cone of B1 leads to significant improvement in bounding the connected correlator as compared to B2. While B1 has a slightly worse time-dependence than B3 (as $0 < \alpha - 2d < 1$), it has a much better r-dependence. And, of course, when taken together, B1 follows a tighter light cone than B3, leading to an overall more useful bound. Thus, while B3 may strictly have a better time-dependence, B1 provides the tightest holistic bound on the growth of connected correlators.

A nearly identical calculation allows us to place stricter bounds on the time required to generate topologically ordered states from topologically trivial ones. We define topologically ordered states as follows: consider a lattice Λ with diameter L and $O(L^d)$ sites. We say that a set of orthonormal states $\{|\psi_1\rangle, \dots, |\psi_k\rangle\}$ are topologically ordered if there exists a constant δ such that

$$\varepsilon \equiv \sup_O \max_{1\leq i,j\leq k} \{ |\langle\psi_i| O |\psi_j\rangle - \langle\psi_j| O |\psi_i\rangle, 2\langle\psi_i| O |\psi_j\rangle| \} \tag{S123}$$

is bounded $\varepsilon = O(L^{-\delta})$. The supremum is taken over operators O supported on a subset of the lattice with diameter $\ell' < L$, so ε essentially measures the ability to distinguish between states $|\psi_i\rangle$ using an operator O supported on only a fraction of the lattice. In contrast, we say the states are topologically trivial if ε is independent of L. Given a set of topologically trivial states $\{|\phi_i\rangle\}_i$ and a set of topologically ordered states $\{|\psi_i\rangle\}_i$, the question is how long it takes to generate a unitary U such that $U|\phi_i\rangle = |\psi_i\rangle$ for all i using a power-law Hamiltonian. Ref. [S4] proves that this time is controlled by the time it takes $\|O(t) - O(t,\ell')\|$ to become non-vanishing in L, where $O(t,\ell')$ is the truncation of the time evolution of O to a radius ℓ'. This expression is bounded in the exact same way as the connected correlator was, and so we see the same improvement from B1 as compared to both B2 and B3.

Finally, we consider the task of simulating the evolution of a local observable under a power-law Hamiltonian H using quantum simulation algorithms. In contrast to the earlier applications, it is not sufficient to simply truncate the time-evolved observable to the light cone. Instead, to simulate the observable, we need to construct the Hamiltonian that generates the dynamics of the observable inside the light cone.

Let A be a unit-norm, single-site observable originally supported on site x, and consider $A(t)$ its evolution under a 2-local power-law Hamiltonian H. Define H_r to be the Hamiltonian constructed by taking terms of H that are fully supported within $\mathcal{B}_r(x)$, and let $\tilde{A}(t)$ be $A(0)$ evolved under H_r (note that this is different than our previous definition of $\tilde{A}$). The question is how large r must be (i.e., how many terms of H must we simulate) for $\left\| A(t) - \tilde{A}(t) \right\|$ to have small error. Intuitively, this observable should be constrained to lie mostly within the light cone of a Lieb-Robinson bound for H as long as the tail of the bound decays sufficiently quickly, so we expect r to be related to the lightcone of our bounds. Refs. [S4, S6] make this intuition rigorous and yield

$$\left\| A(t) - \tilde{A}(t) \right\| \lesssim \frac{t^{\gamma'}}{r^{\beta'}}, \tag{S124}$$

where γ' and β' are listed in Table S1. In particular, in order to ensure only a constant error, we must choose $r \sim t^{\gamma'/\beta'}$, which corresponds to simulating about $r^2 \sim t^{2d\gamma'/\beta'}$ terms of the Hamiltonian. We can compare this exponent $\phi \equiv \gamma'/\beta'$ between bounds:

$$\phi_{\text{B1}} - \phi_{\text{B2}} = -\frac{(\alpha-1)(\alpha-d)+\alpha}{(\alpha-d)(\alpha-2d)}, \tag{S125}$$

$$\phi_{\text{B1}} - \phi_{\text{B3}} = -\frac{(\alpha-d)^2+d}{(\alpha-2d)(\alpha-d)}. \tag{S126}$$

These differences are all negative for $2d < \alpha < 2d+1$, meaning the current work provides the tightest bound on how many terms must be kept to get constant error when simulating the evolution of local observables in this regime.

It would be also interesting to study whether these tighter light cones help generalize other applications of the Lieb-Robinson bounds, such as the entanglement area law [S8] and the Lieb-Schultz-Mattis theorem [S9], to power-law interactions.

S3. A SUMMARY OF THE LIEB-ROBINSON BOUNDS FOR POWER-LAW INTERACTIONS

We provide in Table S2 a brief summary of several Lieb-Robinson bounds for power-law interactions, including the bound presented in the main text, and the saturating protocols.

Regime	Bound		Light cone	Saturating protocol	
$\alpha > 2d+1$	$C(t,r) \lesssim t^{d+1}/r^{\alpha-d}$	Refs.[S4, S10]	$t \gtrsim r$	$t = \mathcal{O}(r)$	—
$\alpha = 2d+1$	$C(t,r) \lesssim \left(t/r^{1-\varepsilon}\right)^{d+1-\varepsilon}$	This paper	$t \gtrsim r^{1-\varepsilon}$	$t = \mathcal{O}(r)$	—
$2d < \alpha < 2d+1$	$C(t,r) \lesssim \left(t/r^{1-\varepsilon}\right)^{\frac{\alpha-d}{\alpha-2d}-\frac{\varepsilon}{2}}$	This paper	$t \gtrsim r^{\alpha-2d-\varepsilon}$	$t = \mathcal{O}(r^{\alpha-2d})$	Ref. [S11]
$\alpha = 2d$	$C(t,r) \lesssim e^{vt}/r^{d}$	Ref. [S2]	$t \gtrsim \log r$	$t = \mathcal{O}(e^{\gamma\sqrt{\log r}})$	Ref. [S11]
$d < \alpha < 2d$	$C(t,r) \lesssim e^{vt}/r^{\alpha-d}$	Ref. [S2]	$t \gtrsim \log r$	$t = \mathcal{O}(\mathrm{polylog}(r))$	Ref. [S11]
$\alpha = d$	$C(t, r \propto N^{1/d}) \lesssim e^{\Theta(\log N)t} - 1$	Ref. [S12]	$t \gtrsim 1/\log N$	$t = \mathcal{O}(\log N)$	Ref. [S13]
$0 \le \alpha < d$	$C(t, r \propto N^{1/d}) \lesssim e^{\Theta(N^{1-\alpha/d})t} - 1$	Ref. [S12]	$t \gtrsim 1/N^{1-\alpha/d}$	$t = \mathcal{O}(1/N^{1-\alpha/d})$	Ref. [S12]

TABLE S2. A summary of the Lieb-Robinson bound on $C(t,r) \coloneqq \left\|\mathbb{P}_r e^{\mathcal{L}t}\left|O\right)\right\|$ for a unit-norm operator O initially supported on a single site. In the last column, we list several protocols for propagating information. Up to subalgebraic corrections, these protocols saturate the corresponding light cones for all $\alpha \ge 0$. The big $\mathcal{O}$ and big Ω are the standard Bachmann-Landau notations. We use $\varepsilon > 0$ to denote arbitrarily small positive constants. In the first two rows ($\alpha \ge 2d+1$), the linear light cone can be saturated by a trivial protocol that consecutively swaps nearest-neighboring spins. In the fourth row, $\gamma = 3\sqrt{d}$ is a constant. For $\alpha \le d$, the bound additionally depends on the total number of sites N in the system. We write the bound at $r \propto N^{1/d}$, which is the largest linear length scale in the system, to simplify the expression.

S4. THE PROOF STRATEGY COMPARED TO PREVIOUS WORKS

In this section, we discuss on a high level the similarities and the differences between our proof strategy and the strategies of previous works.

The proof strategy that involves breaking up the interactions into different length scales dates back to the first polynomial light cone by Foss-Feig et al. [S5]. This strategy was also employed in the more recent proofs of the linear light cones for $\alpha > 2d+1$ by Chen and Lucas [S1] and Kuwahara and Saito [S10]. In particular, Ref. [S10] and this manuscript both use the interaction-picture technique introduced in Ref. [S5] to recursively add longer- and longer-range interactions to existing bounds. On a high level, the key difference between these works is in how the length scales are chosen.

In Ref. [S5], the authors simply divided the interactions into two length scales, resulting in an exponentially tighter (but still not optimal) light cone for $\alpha > 2d$ compared to the previous work by Hasting and Koma [S2]. Instead of two, Ref. [S1] divided the interactions into multiple length scales proportional to 2^k for integer values of k, proving the first linear light cone for $\alpha > 2d$ in $d = 1$ dimension. Ref. [S10] later provided a proof of this linear light cone for all $d \ge 1$, but using doubly exponentially growing length scales. These length scales would not have produced the desired light cone $t \gtrsim r^{\alpha-2d}$ for $\alpha \in (2d, 2d+1)$. For example, if we were to simply extrapolate the bound in Eq. (7) of Ref. [S10] to $\alpha \in (2d, 2d+1)$, this would have resulted in a light cone $t \gtrsim r^{(\alpha-d)/(d+1)}$. This hypothetical light cone is tighter than the one we expect and is violated by the protocol of Ref. [S11], suggesting a proof based on the length scales of Ref. [S10] would have broken down for $\alpha \in (2d, 2d+1)$.

Our strategy for breaking up the interactions bears more resemblance to Ref. [S1] than to Ref. [S10] in that the length scales grow exponentially as L^k for some L. However, the intuition from the recent optimal protocol in Ref. [S11] suggests that L should depend on α for $\alpha \in (2d, 2d+1)$. Another key insight that inspired our choice of L is a guess on what the optimal bound should be. Recall that we were looking for an upper bound on how much an initially local operator "spreads" beyond distance r at time t. Perturbatively at small time, the operator can spread beyond r using all interactions that couple the origin and sites i such that $\mathrm{dist}(i,0) \ge r$. The maximum total strength of such interactions is roughly

$$\sum_{i:\mathrm{dist}i,0\geq r} \frac{1}{\mathrm{dist}(i,0)^\alpha} \sim \frac{1}{r^{\alpha-d}}. \tag{S127}$$

Any hypothetical Lieb-Robinson bound that, at fixed time, decays with distance faster than $1/r^{\alpha-d}$ would be violated by this simple protocol. We conjectured that this so-called "tail" of the optimal bound must be exactly $1/r^{\alpha-d}$. This conjecture, together with the desired light cone $t \gtrsim r^{\alpha-2d}$, suggests the bound

$$\left\|\mathbb{P}_r e^{\mathcal{L}t}|O)\right\| \lesssim \left(\frac{t}{r^{\alpha-2d}}\right)^{\frac{\alpha-d}{\alpha-2d}}. \tag{S128}$$

This nontrivial guess guided us in choosing the length scales, which eventually lead to the bound in Theorem 1 in the main text.

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