Supplemental Material

In this supplemental material, we compute the dissipative part of Hall conductivity using the Kubo formula (Sec. S.I), derive an expression for the flux of angular momentum of the electromagnetic field (Sec. S.II), construct the Keldysh action of a TI out of thermal equilibrium with the environment (Sec. S.III), and finally evaluate the radiation of energy and angular momentum on the basis of the Keldysh formalism (Sec. S.IV).

S.I. Dissipative Hall conductivity

The surface of a topological insulator is described by the Hamiltonian in Eq. (1) of the manuscript,

$$H = (-1)^L v(\sigma_x k_x + \sigma_y k_y) + \sigma_z \Delta, \tag{S1}$$

where $L = 0, 1$ refers to the top and bottom surface, respectively. Despite the minus sign, the conductivity on the top and bottom surface is identical. The spectrum of this Hamiltonian is given by

$$\varepsilon_{\mathbf{k}\,\mathrm{c/v}} = \pm\sqrt{v^2\mathbf{k}^2 + \Delta^2}, \tag{S2}$$

where c (v) denotes the conduction (valence) band. Thus the two bands are separated by a finite energy gap, 2Δ. Conductivity can be computed via the Kubo formula

$$\sigma_{\alpha\beta}(\omega) = \sum_{\mathbf{k}}\sum_{nn'} \frac{f_{\mathbf{k}n} - f_{\mathbf{k}n'}}{\varepsilon_{\mathbf{k}n} - \varepsilon_{\mathbf{k}n'}} \frac{\langle \mathbf{k}n|j_\alpha|\mathbf{k}n'\rangle\langle \mathbf{k}n'|j_\beta|\mathbf{k}n\rangle}{\omega + \varepsilon_{\mathbf{k}n} - \varepsilon_{\mathbf{k}n'} + i\gamma}.$$

The current is given by $j_\alpha = \partial H/\partial k_\alpha = ev\sigma_\alpha$, where the index $\alpha = 1, 2$. To describe the eigenstates, we adopt a notation that $n = 1$ for the conduction band and $n = -1$ for the valence band. Then $\varepsilon_{n\mathbf{k}} = n\varepsilon_{\mathbf{k}}$, where $\varepsilon_{\mathbf{k}} = \sqrt{v^2\mathbf{k}^2 + \Delta^2}$, and the eigenstates are given by [S1]

$$|\mathbf{k}n\rangle = \begin{pmatrix} C_{\uparrow\mathbf{k}n} \\ C_{\downarrow\mathbf{k}n} e^{i\phi_{\mathbf{k}}} \end{pmatrix}, \tag{S3}$$

with $\phi_{\mathbf{k}}$ the azimuthal angle of the momentum, and

$$C_{\uparrow\mathbf{k}n} = \ \mathrm{sgn}(n)\sqrt{\frac{\varepsilon_{\mathbf{k}} + \ \mathrm{sgn}(n)\Delta}{2\varepsilon_{\mathbf{k}}}}, \qquad C_{\downarrow\mathbf{k}n} = \sqrt{\frac{\varepsilon_{\mathbf{k}} - \ \mathrm{sgn}(n)\Delta}{2\varepsilon_{\mathbf{k}}}}. \tag{S4}$$

Given the full spectrum, we can compute the response of the topological material and specifically its characteristic Hall conductivity. At zero temperature, the (ac) Hall conductivity has been computed in Ref. [S1]. At finite temperature, we find the ac Hall conductivity (in units of $e^2/\hbar$) to be

$$\sigma_{xy}(\omega) = \frac{\Delta}{\pi}\int_{\Delta}^{E_c} d\varepsilon \frac{\tanh\left[\varepsilon/2T\right]}{4\varepsilon^2 + (\gamma - i\omega)^2}, \tag{S5}$$

where E_c is the energy cutoff of the Dirac Hamiltonian, which should be associated with the separation of the Dirac point and the closest bulk band [S1]. The latter is roughly of the same order as the bulk band gap E_g, that is, $E_c \sim E_g$. In the limit of zero temperature, we recover the Hall conductivity at zero frequency $\sigma_{xy}(\omega = 0) = 1/(4\pi)$.

In the limit $\gamma \to 0$, we can find an exact expression for the imaginary part of the Hall conductivity:

$$\mathrm{Im}\,\sigma_{xy} = \frac{\Delta}{4\omega}\tanh(\omega/4T)\,\Theta(\omega - 2\Delta). \tag{S6}$$

The Heaviside step function indicates that the imaginary part of the Hall conductivity is nonzero only when an interband transition between the conduction and valence bands is energetically allowed. To obtain the last equation, we have taken the limit $E_c \to \infty$ appropriate at low temperatures $T \ll E_c$. Also, we quote that a similar analysis in the limit $\gamma \to 0$ yields the dissipative part of the ac longitudinal conductivity as

$$\mathrm{Re}\,\sigma_{xx} = \frac{\omega^2 + 4\Delta^2}{16\omega^2}\tanh(\omega/4T)\,\Theta(\omega - 2\Delta). \tag{S7}$$

S.II. Angular momentum of the electromagnetic field

The angular momentum of electromagnetic fields in vacuum (in Gaussian units and choosing $c = 1$) is given by the standard formula where the role of momentum is played by the Poynting vector as the local energy current. For our purposes, it is more convenient to express the angular momentum in terms of the energy-momentum tensor $T^{\mu\nu}$. To this end, we consider the tensor [S2]

$$L^{\mu\nu} = \int \left(x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}\right) dS_\lambda, \tag{S8}$$

where the Greek indices run over $\{0, 1, 2, 3\}$ with 0 the time coordinate, and the "surface" integral is over a three-dimensional hypersurface in space-time. The angular momentum is given by the spatial components L^{ij} with $i, j = 1, 2, 3$, and with the choice of the hypersurface as a "time slice," i.e., the entire space at a given time. The angular momentum vector is simply given by $L_i = (1/2)\epsilon_{ijl}L^{jl}$ with ϵ_{ijl} the completely antisymmetric tensor. Defining the integrand of Eq. (S8) as $m^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$, we find the continuity equation (summation over repeated indices is assumed)

$$\frac{\partial}{\partial x^\lambda} m^{\mu\nu\lambda} = 0. \tag{S9}$$

Choosing $\mu = i$ and $\nu = j$ as spatial indices, we have $\partial_t m^{ij0} + \partial_l m^{ijl} = 0$. Hence, the rate of angular momentum change contained in a given spatial region is obtained from the flux of a rank-3 tensor as

$$\frac{d}{dt} L^{ij} = \int m^{ijl} d\Sigma_l, \tag{S10}$$

where the integral is over a closed two-dimensional surface enclosing the spatial region (the normal to the surface is chosen to be outward). Specifically, we are interested in the radiation of the z-component of angular momentum (hence, $i = x$ and $j = y$) along the z direction (hence, $l = z$); this quantity is given by

$$N_z = \int m^{xyz} dxdy = \int \left(xT^{yz} - yT^{xz}\right) dxdy = -\frac{1}{4\pi} \int \left[(xE_y - yE_x)E_z + (xB_y - yB_x)B_z\right] dxdy, \tag{S11}$$

where, in the last step, we have identified the spatial components of the energy-momentum tensor as the negative Maxwell stress tensor $T^{ij} = -\sigma_{ij}$ and used the fact that the off-diagonal elements of the stress tensor are given by $\sigma_{ij} = \frac{1}{4\pi}(E_i E_j + B_i B_j)$ for $i \neq j$.

S.III. Keldysh action of the TI

In a gauge where the scalar potential is zero, the action describing the interaction with the surface of the TI is given by

$$S_{\mathrm{TI}} = \int_0^\infty \frac{d\omega}{2\pi} \int_{\mathrm{TI}} \begin{pmatrix} A_x^{cl*} & A_y^{cl*} & A_x^{q*} & A_y^{q*} \end{pmatrix} \left(\begin{array}{c|c} \mathbf{0} & \mathbf{\Pi}^A(\omega) \\ \hline \mathbf{\Pi}^R(\omega) & \mathbf{\Pi}^K(\omega) \end{array}\right) \begin{pmatrix} A_x^{cl} \\ A_y^{cl} \\ A_x^q \\ A_y^q \end{pmatrix}, \tag{S12}$$

where $\mathbf{\Pi}^{R/A/K}$ represent 2×2 retarded, advanced, and Keldysh components of the current-current correlation matrix. The retarded and advanced components are given by [S3]

$$\mathbf{\Pi}^R \equiv \mathbf{\Pi} = \begin{pmatrix} \Pi_{xx} & \Pi_{xy} \\ -\Pi_{xy} & \Pi_{xx} \end{pmatrix}, \qquad \mathbf{\Pi}^A = (\mathbf{\Pi}^R)^\dagger = \begin{pmatrix} \Pi_{xx}^* & -\Pi_{xy}^* \\ \Pi_{xy}^* & \Pi_{xx}^* \end{pmatrix}, \tag{S13}$$

while the Keldysh component is dictated by the fluctuation-dissipation relation [S4]

$$\mathbf{\Pi}^K(\omega) = (2f(\omega, T) + 1)\left[\mathbf{\Pi}^R(\omega) - \mathbf{\Pi}^A(\omega)\right] = i(4f(\omega, T) + 2) \begin{pmatrix} \mathrm{Im}\,\Pi_{xx} & -i\mathrm{Re}\,\Pi_{xy} \\ i\mathrm{Re}\,\Pi_{xy} & \mathrm{Im}\,\Pi_{xx} \end{pmatrix}. \tag{S14}$$

Given the symmetries of the correlation matrix, we can write the action in the basis defined by $A_{\pm}^{cl/q} = \big(A_x^{cl/q} \pm iA_y^{cl/q}\big)/\sqrt{2}$ as

$$S_{\rm TI} = \sum_{s=\pm} \int \frac{d\omega}{2\pi} \int_{\rm TI} \begin{pmatrix} A_s^{cl*} & A_s^{q*} \end{pmatrix} \begin{pmatrix} 0 & \Pi_s^A \\ \Pi_s^R & \Pi_s^K \end{pmatrix} \begin{pmatrix} A_s^{cl} \\ A_s^q \end{pmatrix}, \tag{S15}$$

where $\Pi_\pm^R = \Pi_\pm^{A^*} = \Pi_{xx} \mp i\Pi_{xy}$ and $\Pi_\pm^K = i(4f(\omega,T)+2){\rm Im}\,\Pi_\pm$.

S.IV. Radiation via the Keldysh action

To compute the radiation, we should compute the expectation value of a certain flux which takes a bilinear form in the vector field. To this end, we first compute the expectation value of $\langle A_i A_j\rangle$. In the Keldysh basis and in the frequency domain, we need to calculate

$$\langle A_i(\mathbf{x},t)A_j(\mathbf{x},t)\rangle = {\rm Re}\int_0^\infty \frac{d\omega}{2\pi}\left\langle A_i^{cl}(\mathbf{x},\omega)A_j^{cl*}(\mathbf{x},\omega)\right\rangle. \tag{S16}$$

To the first order in α, we must compute

$$\begin{aligned} C_{ij}(\mathbf{x},\omega) &\equiv \left\langle A_i^{cl}(\mathbf{x},\omega)A_j^{cl*}(\mathbf{x},\omega)\right\rangle \\ &= \left\langle A_i^{cl}(\mathbf{x},\omega)A_j^{cl*}(\mathbf{x},\omega)\right\rangle_0 + i\left\langle A_i^{cl}(\mathbf{x},\omega)A_j^{cl*}(\mathbf{x},\omega)\int_{\rm TI}\begin{pmatrix} A_\pm^{cl*} & A_\pm^{q*}\end{pmatrix}\begin{pmatrix} 0 & \Pi_\pm^A \\ \Pi_\pm^R & \Pi_\pm^K\end{pmatrix}\begin{pmatrix} A_\pm^{cl} \\ A_\pm^q\end{pmatrix}\right\rangle_0 + \cdots, \end{aligned} \tag{S17}$$

where the subscript 0 indicates that the expectation value is weighted by the electromagnetic fluctuations in vacuum.

We are only interested in the radiation part of the correlation function. Therefore, we write the correlation function as

$$C_{ij}(\mathbf{x},\omega) = C_{ij}^{\rm eq}(\mathbf{x},\omega) + 4\left[f(\omega,T) - f(\omega,T_{\rm env})\right]{\rm Im}\,\Pi_\pm^R \int_{\mathbf{x}'\in{\rm TI}} \langle A_i^{cl}(\mathbf{x},\omega)A_\pm^{q*}(\mathbf{x}',\omega)\rangle_0 \langle A_j^{cl*}(\mathbf{x},\omega)A_\pm^{q}(\mathbf{x}',\omega)\rangle_0 + \cdots, \tag{S18}$$

where the first term indicates the *equilibrium* correlation function when the TI and the environment are in thermal equilibrium at a global temperature $T_{\rm env}$.[1] This term does not contribute to the net radiation. To evaluate the last term, we need the retarded/advanced free Green's functions

$$\begin{aligned} \langle A_i^{cl}(\mathbf{x},\omega)A_j^{q*}(\mathbf{x}',\omega)\rangle_0 &= iG_{ij}^R(\mathbf{x}-\mathbf{x}',\omega), \\ \langle A_i^{cl*}(\mathbf{x},\omega)A_j^{q}(\mathbf{x}',\omega)\rangle_0 &= iG_{ij}^A(\mathbf{x}-\mathbf{x}',\omega). \end{aligned} \tag{S19}$$

For the electromagnetic field in free space, we have $\mathbb{G}^R = \mathbb{G}^{A*} = (\mathbb{I} - \omega^{-2}\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}')e^{i\omega|\mathbf{x}-\mathbf{x}'|}/|\mathbf{x}-\mathbf{x}'|$.

For future convenience, we introduce a convenient notation. Consider a bilinear hermitian operator $O = X(\mathbf{x},t)Y(\mathbf{x},t)$ in the vector field, where both X and Y are linear in the vector field and may involve derivatives. We then define

$$\langle\langle O\rangle\rangle_\pm \equiv {\rm Re}\left[\langle X^{cl}(\mathbf{x},\omega)A_\pm^{q*}(\mathbf{x}',\omega)\rangle_0\langle Y^{cl*}(\mathbf{x},\omega)A_\pm^{q}(\mathbf{x}',\omega)\rangle_0\right]. \tag{S20}$$

Intuitively, this expression gives the expectation value of the operator defined at $\mathbf{x}$ conditioned on (right/left) circularly polarized sources at the point $\mathbf{x}'$. This definition can be generalized to any linear combination of bilinear operators $O = \sum_{\alpha\beta}X_\alpha Y_\beta$ or to a spatial integral $\int_{\mathbf{x}} XY$.

To make good use of the notation introduced above, we consider the total flux of energy across a plane at a fixed $z > 0$,

$$S_z = \frac{1}{4\pi}\int_{\rm surface}(\mathbf{E}\times\mathbf{B})_z, \tag{S21}$$

[1] The conductivity that appears in $C^{\rm eq}$ is assumed to be 'frozen' at the actual temperature of the TI.

as well as the total flux of angular momentum along the z direction,

$$N_z = -\frac{1}{4\pi}\int dxdy\,[(\mathbf{x}\times\mathbf{E})_z E_z + (\mathbf{x}\times\mathbf{B})_z B_z]. \tag{S22}$$

Some algebra using the retarded and advanced Green's functions yields

$$\langle\langle S_z\rangle\rangle_\pm = \frac{\omega^2}{3}, \tag{S23}$$

while

$$\langle\langle N_z\rangle\rangle_\pm = \pm\frac{\omega}{3}. \tag{S24}$$

Interestingly, we find that

$$\frac{\langle\langle S_z\rangle\rangle_\pm}{\langle\langle N_z\rangle\rangle_\pm} = \pm\omega. \tag{S25}$$

This result indicates that the radiation is emitted in quanta of light, where the ratio of energy ($\hbar\omega$) to angular momentum given be circular polarization ($\pm\hbar$) is given by $\pm\omega$; see the discussion on page 350 of Ref. [S5].

Utilizing the notation introduced above in Eq. (S18), the total energy radiation is given by

$$\langle S_z\rangle = \frac{2A}{\pi}\sum_\pm\int_0^\infty d\omega\big[f(\omega,T) - f(\omega,T_{\rm env})\big]\,\mathrm{Im}\,\Pi^R_\pm\;\langle\langle S_z\rangle\rangle_\pm, \tag{S26}$$

where A is the area of the TI. Using Eq. (S23), we find that the energy radiation depends on $\mathrm{Im}\,(\Pi_+ + \Pi_-) = 4\omega\mathrm{Re}\,\sigma_{xx}$ with an additional factor of 2 included due to the contribution of both top and bottom surfaces. The total energy radiation including a multiplicative factor of 2 accounting for the radiation to both $z\to\pm\infty$ is given by

$$\text{Energy radiation} = \frac{16\hbar A}{3\pi c^2}\int_0^\infty d\omega\,\frac{\omega^3}{e^{\hbar\omega/T}-1}\,\mathrm{Re}\;\sigma_{xx}. \tag{S27}$$

We thus find that, to this order ($\sim\alpha$), the energy radiation is determined by the longitudinal conductivity σ_{xx}.

We can similarly obtain the radiation of angular momentum as

$$\langle N_z\rangle = \frac{2A}{\pi}\sum_\pm\int_0^\infty d\omega\big[f(\omega,T) - f(\omega,T_{\rm env})\big]\,\mathrm{Im}\,\Pi^R_\pm\;\langle\langle N_z\rangle\rangle_\pm. \tag{S28}$$

Using Eq. (S24), we then find that the energy radiation depends on $\mathrm{Im}\,(\Pi^R_+ - \Pi^R_-) = 4\omega\mathrm{Im}\,\sigma_{xy}$; an overall factor of 2 is due to the contribution of both top and bottom surfaces. The total angular momentum radiation including a multiplicative factor of 2 accounting for the radiation to both $z\to\pm\infty$ is then given by

$$\text{Angular momentum radiation} = \langle N_z^{\rm tot}\rangle = 2\langle N_z\rangle = -\frac{16\hbar A}{3\pi c^3}\int_0^\infty d\omega\,\frac{\omega^2}{e^{\hbar\omega/T}-1}\,\mathrm{Im}\;\sigma_{xy}. \tag{S29}$$

With the analytical expression (S6) in the limit $\gamma\to 0$, we find the analytically exact solution

$$\langle N_z^{\rm tot}\rangle = -\frac{\hbar\alpha A\Delta^3}{c^2}\,g\left(\frac{T}{\hbar\Delta}\right), \tag{S30}$$

where

$$g(x) = \frac{4}{3\pi}\int_2^\infty dz\,\frac{z}{(1+\exp[z/(2x)])^2}. \tag{S31}$$

This integral can be computed analytically; it will be more convenient to define the function in terms of the inverse argument $y = x^{-1}$:

$$g(x=y^{-1}) = \frac{16}{3\pi y^2}\left[\frac{\pi^2}{6} + \mathrm{Li}_2\left(-e^y\right) - \frac{1}{2}y^2 - \frac{y}{e^{-y}+1} + (y-1)\log\left(e^y+1\right)\right], \tag{S32}$$

where Li_2 is the polylogarithm of order 2. At large $x\gg 1$, we have $g(x)\sim ax^2$ with $a = 4(\pi^2 - 12\log 2)/(9\pi)\approx 0.22$. In this limit, angular momentum radiation scales quadratically with temperature. At low temperatures, the radiation is exponentially suppressed in the gap size $\sim\exp(-2\hbar\Delta/T)$.

A. Polarization

To quantify the degree of circular polarization of the emitted radiation, we define

$$\text{Circular Pol.} \equiv \frac{\text{Angular momentum radiation}/\hbar}{\#\text{ radiated photons}}, \tag{S33}$$

where the denominator is given by the total number of radiated photons per unit time. For a completely clockwise/counterclockwise circularly polarized light (defined with respect to the positive z direction), this quantity is ± 1. From Eq. (S27), we have (each photon contributes to the energy radiation by $\hbar\omega$; the total number of radiated photons is obtained by dividing the integrand in Eq. (S27) by $\hbar\omega$)

$$\#\text{ radiated photons} = \frac{16A}{3\pi c^2}\int_0^\infty d\omega\, \frac{\omega^2}{e^{\hbar\omega/T}-1}\,\text{Re}\,\sigma_{xx}. \tag{S34}$$

With the longitudinal conductivity in Eq. (S7), we find

$$\#\text{ radiated photons} = \frac{\alpha A\Delta^3}{c^2}\, h\left(\frac{T}{\hbar\Delta}\right), \tag{S35}$$

where

$$h(x) = \frac{1}{3\pi}\int_2^\infty dz\, \frac{z^2+4}{(1+\exp[z/(2x)])^2}\,. \tag{S36}$$

This integral can be computed analytically; it will be more convenient to define the function h in terms of the inverse argument $y = x^{-1}$:

$$h(x=y^{-1}) = \frac{8}{9\pi y^3}\left[6(y-1)\text{Li}_2\left(-e^y\right) - 6\text{Li}_3\left(-e^y\right) - 4y^3 + 3y^2\tanh\left(\frac{y}{2}\right) + 6(y-1)y\log\left(e^y+1\right) - \pi^2\right], \tag{S37}$$

where Li_3 is the polylogarithm of order 3. The degree of circular polarization is then given by

$$\text{Circular Pol.} = -\frac{g(T/\hbar\Delta)}{h(T/\hbar\Delta)} \tag{S38}$$

We have plotted this quantity (conveniently multiplied by a sign) as a function of temperature in Fig. S1. The (negative) polarization starts at 1, indicating that at low temperatures the radiation is fully circularly polarized. This is because, at low temperatures, the radiation is dominated by the contribution from the top (bottom) of the valence (conduction) band. What is remarkable is that the polarization remains of order 1 even for $\frac{T}{\hbar\Delta} \gtrsim 1$, reflecting the strong spin-orbit coupling of the TI.

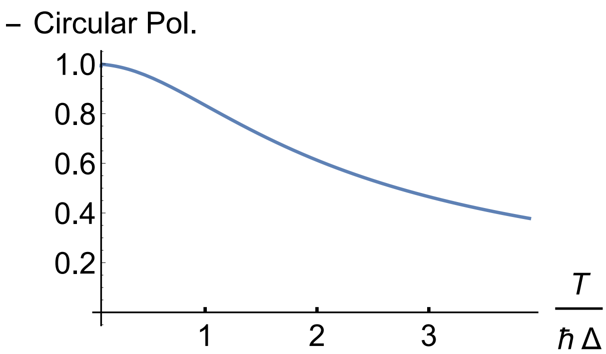

Figure S1: The degree of circular polarization from the TI. The radiation is highly circularly polarized even at temperatures comparable to the gap.

Gyrotropic materials

We contrast our results with the behavior of a gyrotropic material in the presence of a constant magnetic field, and argue the degree of circular polarization (Circular Pol.) in such materials is quite small. The dielectric function for a gyrotropic material is given by (see p. 106 of [S6])

$$\sigma_{xx}(\omega) = i\frac{\omega_p^2}{4\pi}\frac{\omega + i\gamma}{(\omega + i\gamma)^2 - \omega_c^2}, \qquad \sigma_{xy}(\omega) = \frac{\omega_p^2}{4\pi}\frac{\omega_c}{(\omega + i\gamma)^2 - \omega_c^2}, \tag{S39}$$

with ω_p the plasma frequency and $\omega_c = eB/m$ the cyclotron frequency where B is the magnetic field along the z direction. Now even a large magnetic field of the order of 1 Tesla gives rise to a cyclotron frequency of the order $\omega_c \sim 10^{-4}$eV$\sim 1°$K. The degree of circular polarization is then of the order (with $B = 1$ Tesla)

$$\text{Circular Pol.} \sim \frac{\omega_c}{T} \sim \frac{1}{T \text{ in } °\text{K}} \lesssim 10^{-2}, \tag{S40}$$

where the numerical estimate is provided at room temperature. In practice, the expected radiation is even smaller as the dissipative part of σ_{xy} is suppressed by a factor of ω_c/γ due to scattering from impurities. This quantity can be measured in quantum oscillation experiments [directly probing scattering time ($\tau = \gamma^{-1}$) and the mean free path ($l = v_F\tau$)], and is typically small.

[S1] W.-K. Tse and A. H. MacDonald, Phys. Rev. B **84**, 205327 (2011).
[S2] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, vol. 2 (Butterworth-Heinemann, 1987).
[S3] G. D. Mahan, *Many-particle physics* (Springer Science & Business Media, 2000).
[S4] A. Kamenev, *Field theory of non-equilibrium systems* (Cambridge University Press, 2011).
[S5] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998), 3rd ed.
[S6] J. A. Bittencourt, *Fundamentals of plasma physics* (Springer Science & Business Media, 2013).