Supplemental Material for "Kramers' degeneracy for open systems in thermal equilibrium"

The Supplemental Material is organized as follows: In Sec. 1, we complete the steps in the proof of the Kramers' theorem for Lindbladians. We also explain why the degeneracy only appears for fermonic systems (as opposed to spin systems) and describe an analogous Kramers' degeneracy for thermal quantum channels. In Sec. 2, we show how the Kramers' degeneracy manifests in the single-particle spectrum of "quadratic Lindbladians". Sec. 3 shows that microreversibility of the Lindbladian arises naturally from TRS-invariant system-bath coupling in the case of a thermal bath. Sec. 4 describes linear response in open quantum systems, suggesting that the degeneracy can be probed via tunneling spectroscopy experiments.

1. KRAMERS' THEOREM AND GREEN'S FUNCTION DEGENERACY

Orthogonal solutions. We complete the steps of the generalized Kramers' theorem outlined in the main text. From the main text, we have seen that if the Lindbladian satisfies a microreversibility condition: $\mathcal{L}_-^\dagger = \mathcal{Q}_-^{-1}\mathcal{T}_-^{-1}\mathcal{L}_-\mathcal{T}_-\mathcal{Q}_-$ with $\mathcal{L}_-(r_i) = \Lambda_i r_i, \mathcal{L}_-^\dagger(l_i) = \Lambda_i^* l_i$, then r_i and $\mathcal{T}\mathcal{Q}(l_i)$ are both right eigenoperators of $\mathcal{L}_-$ with eigenvalue Λ_i. We now proceed to show that these are indeed orthogonal solutions by showing that $\mathrm{Tr}[l_i^\dagger \mathcal{T}\mathcal{Q}(l_i)] = 0$:

$$\mathrm{Tr}[l_i^\dagger T(ql_i)T^{-1}] = \mathrm{Tr}[(T(T(ql_i)T^{-1})T^{-1})^\dagger T l_i T^{-1}] \tag{S1}$$
$$= \mathrm{Tr}[(T^2(ql_i)T^{-2})^\dagger T l_i T^{-1}] \tag{S2}$$
$$= -\,\mathrm{Tr}[(ql_i)^\dagger T l_i T^{-1}] \tag{S3}$$
$$= -\,\mathrm{Tr}[l_i^\dagger q T l_i T^{-1}] \tag{S4}$$
$$= -\,\mathrm{Tr}[l_i^\dagger T(ql_i)T^{-1}] \tag{S5}$$
$$= 0. \tag{S6}$$

In Eq. (S1), we have used the relation:

$$\mathrm{Tr}[(T\psi T^{-1})^\dagger (T\phi T^{-1})] = \mathrm{Tr}[(U\psi^* U^\dagger)^\dagger (U\phi^* U^\dagger)] \tag{S7}$$
$$= \mathrm{Tr}[U\psi^T U^\dagger U\phi^* U^\dagger] \tag{S8}$$
$$= \mathrm{Tr}[\psi^T \phi^*] \tag{S9}$$
$$= \mathrm{Tr}[\phi^\dagger \psi], \tag{S10}$$

for any ψ, ϕ, where $T = UK$. In Eq. (S4), we have used $\mathcal{T}^2(ql_i) = q\mathcal{T}^2(l_i) = -ql_i$, i.e. l_i is an eigenoperator of the odd-superparity sector of the Lindbladian.

Definition of the Green's function. Here we highlight an important subtlety regarding the definition of the Green's function Eq. (13), and in particular the interpretation of the time-evolved fermion operator $\hat{f}_{i,\sigma}(t)$.

A general Green's function of two observables $G_{AB}(t) = \mathrm{Tr}(A(t)B\rho_{\mathrm{SS}})$ describes the influence of a perturbation at time 0 on the outcome of a measurement at time t. Since fermion parity symmetry is fundamental, it is not possible to perturb the system by a fermion-parity-odd operator such as $f_{i,\sigma}^\dagger$; rather, in any physical protocol where the Green's function is measured, the initial perturbation will involve some exchange of fermions between the system and some probe. Thus, we should understand that the operator B is a product of $\hat{f}_{i,\sigma}$ with some fermionic operator acting on the probe. (See Section 4 of the SM for an example of this construction in the context of tunneling spectroscopy.) We heuristically write $B \sim f_{i,\sigma}^\dagger f_{\mathrm{pr}}$, where f_{probe} is an unspecified fermionic operator acting on the probe, ensuring that B itself is a superparity-even operator. Similarly, the observable A that we measure at time t will be a product of fermion-superparity-odd operators on the system and probe: $A \sim f_{i,\sigma} f'_{\mathrm{pr}}$ for some other probe operator f'_{pr}.

With this in mind, the Heisenberg picture evolution of the observable A is given by $A(t) = e^{\mathcal{L}^\dagger t}[A]$, where $\mathcal{L}^\dagger$ is defined in Eq. (6). The Lindbladian only involves operators acting on the system, and not the probe. Therefore, when considering the observable $A = f_{i,\sigma} f'_{\mathrm{pr}}$, it is tempting to pull the probe operator f'_{pr} outside the evolution superoperator, so that it can combine with the other probe operator in B. Indeed, this is precisely what is done when calculating fermionic Green's functions in closed systems where $\mathcal{L} = -i[H, \cdot]$. However, when the system is open, it is possible for fermions to move between the system and the environment (not to be confused with the probe), which leads to jump operators L_i that are superparity odd. In this case, we cannot ignore the presence of the probe operators, because $\mathcal{L}^\dagger[A]$ will include a term $L_i^\dagger f_{i,\sigma} f'_{\rm pr} L_i$, which differs from the naïve expression $(L_i^\dagger f_{i,\sigma} L_i) f'_{\rm pr}$ by a minus sign. In other words, even though $f'_{\rm pr}$ is a probe operator, its anticommutation with the jump operators means that the evolution of the product $f_{i,\sigma} f'_{\rm pr}$ doesn't factorize as the product of time-evolved operators $f_{i,\sigma}(t)$ and $f'_{\rm pr}$.

This can be remedied by defining a 'dummy' Majorana fermion operator $\eta_{\rm d}$, having no dynamics of its own, and including it such that the correct anticommutation relations are obeyed. Specifically, one can verify that $\mathcal{L}^\dagger[f_{i,\sigma} f'_{\rm pr}] = \eta_{\rm d}\, \mathcal{L}^\dagger[\eta_{\rm d} f_{i,\sigma}] f'_{\rm pr}$, since $\eta_{\rm d}^2 = 1$. This allows probe operators to be pulled out of the system evolution superoperator, such that Green's functions can be defined using operators that pertain to the system only. Specifically, the physically meaningful definition of the time-evolved operator $f_{i,\sigma}(t)$ appearing in Eq. (13) should be:

$$f_{i,\sigma}(t) \coloneqq \eta_{\rm d}\, e^{\mathcal{L}^\dagger t}[\eta_{\rm d} f_{i,\sigma}]. \tag{S11}$$

In practice, we will not explicitly write out the dummy Majorana fermion, but instead understand that it is implicitly included in any expression of the form $e^{\mathcal{L}^\dagger t}[f_{i,\sigma}]$. Alternatively, we can modify all the fermion-superparity-odd jump operators by $L_i \to \eta_{\rm d} L_i$, in which case the usual expression $e^{\mathcal{L}^\dagger t}[f_{i,\sigma}]$ can be used without modification.

Degenerate Green's functions. Having dealt with the above issue, we now show that steady-state Green's functions corresponding to fermions with opposite spin are related in a simple way due to microreversibility. We first consider the quantity

$$\begin{aligned}
\mathrm{Tr}[e^{\mathcal{L}^\dagger t}(f_{i,\sigma}) f_{j,\tau}^\dagger q] &= \mathrm{Tr}[\mathcal{Q}(e^{\mathcal{L}^\dagger t}(f_{i,\sigma}) f_{j,\tau}^\dagger)] && \text{(S12)}\\
&= \mathrm{Tr}[\mathcal{T}\mathcal{Q}(e^{\mathcal{L}^\dagger t}(f_{i,\sigma}) f_{j,\tau}^\dagger)]^* && \text{(S13)}\\
&= \mathrm{Tr}[\mathcal{T}\mathcal{Q}(e^{\mathcal{L}^\dagger t}(f_{i,\sigma}))\mathcal{T}(f_{j,\tau}^\dagger)]^* && \text{(S14)}\\
&= \mathrm{Tr}[e^{\mathcal{L} t}(\mathcal{T}\mathcal{Q}(f_{i,\sigma}))\mathcal{T}(f_{j,\tau}^\dagger)]^* && \text{(S15)}\\
&= \sigma\tau\, \mathrm{Tr}[e^{\mathcal{L} t}(q f_{i,-\sigma}) f_{j,-\tau}^\dagger]^* && \text{(S16)}\\
&= \sigma\tau\, \mathrm{Tr}[(q f_{i,-\sigma}) e^{\mathcal{L}^\dagger t}(f_{j,-\tau}^\dagger)]^*, && \text{(S17)}
\end{aligned}$$

where we have used $\mathrm{Tr}[A] = \mathrm{Tr}[\mathcal{T}(A)]^*$ in (S13), $\mathcal{T}[AB] = \mathcal{T}[A]\mathcal{T}[B]$ in (S14), the definition of microreversibility in (S15), $\mathcal{T}[f_{i,\sigma}] = \sigma f_{i,-\sigma}$ in (S16), and $\mathrm{Tr}[A e^{\mathcal{L}t}(B)] = \mathrm{Tr}[e^{\mathcal{L}^\dagger t}(A) B]$ in (S17).

Let us now define the following generalizations of retarded Green's functions:

$$\begin{aligned}
G_{i\sigma;j\tau} &\equiv -i\Theta(t)\left(\mathrm{Tr}[e^{\mathcal{L}^\dagger t}(f_{i,\sigma}) f_{j,\tau}^\dagger q] + \mathrm{Tr}[f_{j,\tau}^\dagger e^{\mathcal{L}^\dagger t}(f_{i,\sigma}) q]\right) && \text{(S18)}\\
&= -i\Theta(t)\left(\sigma\tau\, \mathrm{Tr}[f_{i,-\sigma} e^{\mathcal{L}^\dagger t}(f_{j,-\tau}^\dagger) q]^* + \sigma\tau\, \mathrm{Tr}[e^{\mathcal{L}^\dagger t}(f_{j,-\tau}^\dagger) f_{i,-\sigma} q]^*\right) && \text{(S19)}\\
&= -i\Theta(t)\left(\sigma\tau\, \mathrm{Tr}[e^{\mathcal{L}^\dagger t}(f_{j,-\tau}) f_{i,-\sigma}^\dagger q] + \sigma\tau\, \mathrm{Tr}[e^{\mathcal{L}^\dagger t}(f_{j,-\tau}) q f_{i,-\sigma}^\dagger]\right) && \text{(S20)}\\
&= -i\Theta(t)\left(\sigma\tau G_{j-\tau;i-\sigma}(t)\right), && \text{(S21)}
\end{aligned}$$

where in (S20) we have used: $\mathrm{Tr}[AB] = \mathrm{Tr}[A^\dagger B^\dagger]^*$ and $[e^{\mathcal{L}^\dagger t}(A)]^\dagger = e^{\mathcal{L}^\dagger t}(A^\dagger)$. So indeed we find that steady-state Green's functions for opposite spin labels are related to each other in a simple way for systems with microreversibility. We have confirmed these expressions numerically for the example system described in the main text.

Non-fermionic systems. As mentioned in the main text, the Kramers' degeneracy of the Lindbladian only arises in fermionic systems, but not in bosonic or spin systems, even though the latter have a Kramers-degenerate Hamiltonian when the total spin is a half-integer. The differences between fermionic vs. non-fermionic open systems becomes apparent when we consider how linear response functions are constrained by microreversibility and TRS, in analogy to Eq. (S21). Regardless of particle statistics, detailed balance implies [Ref. [1], Eq. (2.2)]:

$$\mathrm{Tr}[A(t) B q] = \mathrm{Tr}[\tilde{B}(t)\tilde{A} q], \tag{S22}$$

where $\tilde{A} \coloneqq \mathcal{T}[A]^\dagger$ and similar for $\tilde{B}$. Suppose for the sake of simplicity that $\beta = 0$ and hence $q \propto \mathbb{I}$. Then, by decomposing A and B in terms of right and left eigenoperators of $\mathcal{L}$, respectively, the left-hand side of the above can be written as a linear combination of terms $\mathrm{Tr}[r_i(t) l_j] = e^{\Lambda_i t}\delta_{ij}$, and thus without loss of generality the condition becomes $\mathrm{Tr}[r_i(t) l_i] = \mathrm{Tr}[\tilde{l}_i(t)\tilde{r}_i]$. (The same arguments can in principle be generalized to $\beta > 0$ by defining a new set of operators l'_i which are left eigenoperators of $\mathcal{L}$ with respect to the non-standard inner product $\langle A, B\rangle_q = \mathrm{Tr}[A^\dagger B q]$.)

In a spin system, we find that $\tilde{l}_i = r_i$, which ensures that the above condition can be satisfied, regardless of whether or not $\mathcal{L}$ possesses any degeneracies. The same relation cannot hold true for fermionic superparity-odd eigenoperators, because it would contradict with Eq. (S6) (when one sets $q \propto \mathbb{I}$). The only way for the Green's function identity to hold is if r_i and $\tilde{l}_i$ are independent eigenoperators, which as we proved above implies that they are degenerate. Thus, fermionic systems differ from spin systems in that they cannot be made to satisfy (S22) without degeneracies in the spectrum of $\mathcal{L}$.

Kramers' degeneracy in quantum channels. We briefly note that a similar Kramers' degeneracy can be found in thermal quantum channels, which can describe the discrete time evolution of a system coupled to a *non-Markovian* bath. Define a quantum channel and its adjoint:

$$\mathcal{E}(x) = \sum_i E_i x E_i^\dagger, \qquad \mathcal{E}^\dagger(x) = \sum_i E_i^\dagger x E_i. \tag{S23}$$

The condition for trace preservation of the channel implies: $\mathcal{E}^\dagger(\mathbb{I}) = \sum_i E_i^\dagger E_i = \mathbb{I}$, which is the only condition that the Kraus operators (E_i) need to obey to be a proper channel. Suppose we further impose microreversibility: $\mathcal{E}^\dagger = \mathcal{Q}^{-1}\mathcal{T}^{-1}\mathcal{E}\mathcal{T}\mathcal{Q}$ where $\mathcal{Q}, \mathcal{T}$ are defined as before. It is easy to show that the thermal state q is a steady state (eigenoperator of $\mathcal{E}$ with eigenvalue 1). For physical fermionic channels, the channel superoperator can be split into even and odd superparity sectors: $\mathcal{E} = \mathrm{Diag}[\mathcal{E}_+, \mathcal{E}_-]$. Our analysis implies that the odd superparity sector $\mathcal{E}_-$ must be twofold degenerate.

As an example, the channel superoperator could represent the completely-positive-trace-preserving map for time-dependent Lindblad evolution: $\mathcal{E} = \exp(\int \mathcal{L}(t)dt)$. If the instantaneous Lindbladians $\mathcal{L}(t)$ obey microreversibility at all times, then so should the channel superoperator $\mathcal{E}$. The odd-superparity sector of $\mathcal{E}$ will then have a twofold degeneracy.

2. KRAMERS' DEGENERACY IN QUADRATIC MODELS

We show that "quadratic Lindbladians" can host a twofold degeneracy in their single-particle spectrum in the absence of microreversibility, as long as the system-environment coupling respects time-reversal symmetry. This is in contrast to the (quartic) models studied in the main text, where spectral splitting can emerge due to a non-equilibrium environment even if all couplings respect TRS. The importance of microreversibility is therefore only apparent in quartic models.

Consider a quadratic Hamiltonian $H = \sum_{ij} H_{ij}\alpha_i\alpha_j$ in the presence of linear dissipators $L_\mu = \sum l_{\mu,i}\alpha_i$, where α_i are Majorana fermions. All terms in the master equation are quadratic in fermion operators, which implies that the Lindbladian can be split into superparity sectors: $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$. Define the superoperators: $e_j^\dagger(\rho) = [\alpha_j\rho + (\mathcal{P}\rho)\alpha_j]/2, e_j(\rho) = [\alpha_j\rho - (\mathcal{P}\rho)\alpha_j]/2$, where α_j are Majoranas, and $\mathcal{P}$ is the parity superoperator. Then we can express $\mathcal{L}_+$ as

$$\mathcal{L}_+ = 4i\sum_{ij}(Z_{ji}e_i^\dagger e_j) + (Y_{ij}e_i^\dagger e_j^\dagger) = \sum_i \epsilon_i \beta_i^\dagger \beta_i', \tag{S24}$$

where $Z = H + i\mathrm{Re}[M]$, $Y = \mathrm{Im}[M]$, $M = l^T l^*$, $\epsilon_i/(4i)$ are the eigenvalues of Z, and $\mathrm{Re}[\epsilon_i] < 0$ [2, 3]. Analogously

$$\mathcal{L}_- = 4i\sum_{ij}(Z_{ji}e_i e_j^\dagger) + (Y_{ij}e_i e_j) = \sum_i \epsilon_i + \sum_i(-\epsilon_i)\eta_i^\dagger\eta_i'. \tag{S25}$$

The first term $\epsilon_m \equiv \sum_i \epsilon_i$ is a negative offset, then excitations have a *positive* real energy. The many-body eigenvalues are thus built from single-particle eigenvalues $\{\epsilon_i\}$.

Consider a spin-1/2 Hamiltonian with TRS:

$$H = \frac{1}{2}(\alpha)^T H\alpha, \qquad \alpha = (a_{1,+}, b_{1,+}, \dots, a_{1,-}, b_{1,-}, \dots)^T, \tag{S26}$$

where a, b are Majoranas which transform via $Ta_{i,\sigma}T^{-1} = \sigma a_{i,-\sigma}, Tb_{i,\sigma}T^{-1} = -\sigma b_{i,-\sigma}$, and $H = UH^*U^\dagger$. We include arbitrary linear dissipators which transform into each other (up to a phase) upon action of the symmetry operator:

$$L_{i,+} = \sqrt{\gamma_i}f_{i,+}, \qquad L_{i,-} = \sqrt{\gamma_i}f_{i,-}, \qquad Tf_{i,+}T^{-1} = f_{i,-}, \qquad Tf_{i,-}T^{-1} = -f_{i,+}. \tag{S27}$$

The dissipators can be expanded in terms of Majoranas: $L_{i,+} = \vec{l}_{i,+} \cdot \alpha, L_{i,-} = \vec{l}_{i,-} \cdot \alpha$, which defines the matrix $M = l^T l^*$. We now show that $M = UM^*U^\dagger$:

$$UM^*U^\dagger = U(l^\dagger l)U^\dagger = \begin{pmatrix} \vec{l}_{i,-} \\ -\vec{l}_{i,+} \\ \vdots \end{pmatrix}^T \begin{pmatrix} \vec{l}_{i,-} \\ -\vec{l}_{i,+} \\ \vdots \end{pmatrix}^* = M, \tag{S28}$$

where we have used $U\vec{l}^\dagger_{i,+} = \vec{l}^T_{i,-}, U\vec{l}^\dagger_{i,-} = -\vec{l}^T_{i,+}$ in the second equality. In the last equality, we have used the fact that the Lindbladian is invariant under a relabeling and a change of sign of the dissipators. This expression implies that the spectral matrix $Z = H + i\mathrm{Re}[M]$ satisfies: $Z = UZ^TU^\dagger, U^2 = -\mathbb{I}$, which ensures that the single-particle spectrum is twofold degenerate. Note that we have not imposed microreversibility (detailed balance) for this result. Quadratic Lindbladians are thus special in the sense that the odd superparity sector can host a degeneracy in the absence of thermal equilibrium (unlike the examples in the main text).

3. MICROREVERSIBILITY FROM TRS-INVARIANT SYSTEM-BATH COUPLING

Here, we demonstrate that the Lindbladian of a Markovian open system will respect the microreversibility condition Eq. (5) if time-reversal symmetry is imposed on the system and bath as a whole. Before any Markovian approximation is made, the system and bath can be described by a Hamiltonian

$$H_{\mathrm{tot}} = H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B + H_{SB}, \tag{S29}$$

where H_S, H_B are the system and bath Hamiltonians, respectively, and H_{SB} couples the two. Without loss of generality, we can decompose the latter as [4]

$$H_{SB} = \sum_\alpha A_\alpha \otimes B_\alpha, \tag{S30}$$

where A_α, B_α are Hermitian matrices.

We suppose that the bath is initialized in a thermal Gibbs state $\rho_B = Z_B^{-1} e^{-\beta H_B}$, where $\beta = 1/T$ is the inverse temperature, and $Z_B = \mathrm{Tr}\, e^{-\beta H_B}$ is the partition function for the bath. Then define two-time correlators $\Gamma_{\alpha\beta}(t) = \mathrm{Tr}[B_\alpha(t)B_\beta(0)\rho_B]$, where $B_\alpha(t) = e^{\mathrm{i}H_B t} B_\alpha e^{-\mathrm{i}H_B t}$. Hermiticity of B_α implies that

$$\Gamma_{\alpha\beta}(t)^* = \Gamma_{\beta\alpha}(-t). \tag{S31}$$

The expectation values $\langle B_\alpha \rangle \coloneqq \mathrm{Tr}\, B_\alpha \rho_B$ can always be made to vanish by replacing $B_\alpha \to B_\alpha - \langle B_\alpha \rangle$, and adding a term A_α to H_S, which does not change H_{tot}. We also define the 'lowering' operators $A_\alpha(\omega)$, which are the components of A_α that decrease the energy of the system by ω. More concretely,

$$A_\alpha(\omega) = \sum_{\epsilon' - \epsilon = \omega} \Pi_\epsilon A_\alpha \Pi_{\epsilon'}, \tag{S32}$$

where Π_ϵ is a projector onto the eigenspace of H_S with eigenvalue ϵ.

In order for the open system to be Markovian, the two-time correlation functions must decay over a timescale τ_{m} that is sufficiently short, and the system-bath coupling H_{SB} is sufficiently weak. If these criteria are met, then one can derive an expression for the Lindbladian in terms of the components of the microscopic Hamiltonian (S29) [4]

$$\mathcal{L}[\rho] = -\mathrm{i}[H_S + H_{LS}, \rho] + \sum_\omega \sum_{\alpha\beta} \tilde{\Gamma}_{\alpha\beta}(\omega) \left(A_\beta(\omega)\rho A_\alpha(\omega)^\dagger - \frac{1}{2}\Big\{ A_\alpha(\omega)^\dagger A_\beta(\omega), \rho \Big\} \right), \tag{S33}$$

where $\tilde{\Gamma}_{\alpha\beta}(\omega) = \int \mathrm{d}t e^{\mathrm{i}\epsilon t} \Gamma_{\alpha\beta}(t)$ is the Fourier transform of the two-time correlation functions, which is a Hermitian matrix due to (S31). Here, we have defined the Lamb shift Hamiltonian $H_{LS} = \sum_{\omega,\alpha,\beta} S_{\alpha\beta}(\omega) A_\alpha(\omega)^\dagger A_\beta(\omega)$, where $S_{\alpha\beta}(\omega) = \int \mathrm{d}t\, \mathrm{sgn}(t) e^{\mathrm{i}\epsilon t} \Gamma_{\alpha\beta}(t)$, which is Hermitian and commutes with H_S.

Now, TRS of the combined system and bath implies that H_S and H_B are each TRS-invariant, and that $\mathcal{T}_{SB}[H_{\mathrm{tot}}] = H_{\mathrm{tot}}$, where the superoperator $\mathcal{T}_{SB}$ acts as $\mathcal{T}_{SB}[O] = (U_S \otimes U_B) O^* (U_S^\dagger \otimes U_B^\dagger)$ for any operator O over the system-bath Hilbert space. The operator U_S is the unitary part of the TRS transformation acting on the system, as appears in

Eq. (2), and similarly U_B is a unitary operator acting on the bath, which we leave unspecified for full generality. In terms of the decomposition (S30), we have

$$U_S[A_\alpha]^* U_S^\dagger = \sum_\beta u_{\alpha\beta} A_\beta \qquad\qquad U_B[B_\alpha]^* U_B^\dagger = \sum_\beta [u^{-1}]_{\beta\alpha} B_\beta \tag{S34}$$

for some matrix u. Hermiticity of A_α, B_α implies that u is a real matrix, and $\mathcal{T}^2 = \mathcal{P}$ implies that $[uu^*]_{\alpha\beta} = p_\alpha \delta_{\alpha\beta}$, where $p_\alpha = +1$ (-1) if A_α is a fermion parity even (odd) operator. The above conditions generalise the notion of 'weak' symmetries described in Ref. [5] to include antiunitary symmetry operations, with the difference that we impose restrictions on the microscopic Hamiltonian, rather than the emergent master equation.

Now, if the bath is in thermal equilibrium at inverse temperature β, then this imposes a Kubo-Martin-Schwinger (KMS) condition (or 'detailed balance') on the spectral functions $\Gamma_{\alpha\beta}(\omega)$ [6]:

$$\tilde{\Gamma}_{\alpha\beta}(-\omega) = e^{-\beta\omega} \tilde{\Gamma}_{\beta\alpha}(\omega). \tag{S35}$$

Furthermore, since H_B is TRS-invariant, we can use the transformation property (S34) to determine how the two-time correlation functions transform under TRS

$$\begin{aligned}
\Gamma_{\alpha\beta}(t) &= Z_B^{-1}\,\mathrm{Tr}\left[e^{\mathrm{i}H_B t} B_\alpha e^{-\mathrm{i}H_B t} B_\beta e^{-\beta H_B}\right] \\
&= Z_B^{-1}\,\mathrm{Tr}\left[U_B \left(e^{\mathrm{i}H_B (t+\mathrm{i}\beta)} B_\alpha e^{-\mathrm{i}H_B t} B_\beta\right)^* U_B^\dagger\right]^* \\
&= Z_B^{-1} \sum_{\gamma\delta} [u^{-1}]_{\gamma\alpha} [u^{-1}]_{\delta\beta}\,\mathrm{Tr}\left[e^{-\mathrm{i}H_B(t-\mathrm{i}\beta)} B_\gamma e^{\mathrm{i}H_B t} B_\delta\right]^* \\
&= Z_B^{-1} \sum_{\gamma\delta} [u^{-1}]_{\gamma\alpha} [u^{-1}]_{\delta\beta}\,\mathrm{Tr}\left[B_\delta e^{-\mathrm{i}H_B t} B_\gamma e^{\mathrm{i}H_B(t+\mathrm{i}\beta)}\right] \\
&= \sum_{\gamma\delta} [u^{-1}]_{\gamma\alpha} [u^{-1}]_{\delta\beta} \Gamma_{\delta\gamma}(t).
\end{aligned} \tag{S36}$$

The above can be Fourier transformed to obtain an analogous condition for $\tilde{\Gamma}_{\alpha\beta}(\omega)$.

With all these identities in hand, we are ready to begin our proof. We start by showing that the Gibbs state for the system $\rho_G = Z_S^{-1} e^{-\beta H_S}$ (where $Z_S = \mathrm{Tr}\, e^{-\beta H_S}$ is the system partition function) is a steady state. We have

$$\begin{aligned}
\mathcal{L}[\rho_G] &= \frac{1}{Z_S} \sum_\omega \sum_{\alpha\beta} \tilde{\Gamma}_{\alpha\beta}(\omega) \left(A_\beta(\omega) e^{-\beta H_S} A_\alpha(\omega)^\dagger - \frac{1}{2}\Big\{A_\alpha(\omega)^\dagger A_\beta(\omega), e^{-\beta H_S}\Big\}\right) \\
&= \frac{1}{Z_S} \sum_\omega \sum_{\alpha\beta} \tilde{\Gamma}_{\alpha\beta}(\omega) A_\beta(\omega) e^{-\beta H_S} A_\alpha(\omega)^\dagger - \frac{1}{2} \tilde{\Gamma}_{\beta\alpha}(\omega) \Big\{A_\beta(-\omega) A_\alpha(-\omega)^\dagger, e^{-\beta H_S}\Big\} \\
&= \frac{1}{Z_S} \sum_\omega \sum_{\alpha\beta} \left(\tilde{\Gamma}_{\alpha\beta}(\omega) e^{-\beta\omega} A_\beta(\omega) A_\alpha(\omega)^\dagger - \tilde{\Gamma}_{\beta\alpha}(\omega) A_\alpha(-\omega) A_\beta(-\omega)^\dagger\right) e^{-\beta H_S} = 0.
\end{aligned} \tag{S37}$$

In the first equality, we use $[H_{LS}, H_S] = 0$. The second equality involves a swap of labels α and β, and uses relation $A_\alpha(-\omega) = A_\alpha(\omega)^\dagger$, which can be verified using (S32). The third equality requires the relation $e^{-\beta H_S} A_\alpha(\omega)^\dagger = A_\alpha(\omega)^\dagger e^{-\beta(H_S+\omega)}$, and finally we replace $\omega \to -\omega$ in the second term and employ Eq. (S35).

Now we verify that Eq. (5) is satisfied. Recall that $\mathcal{Q}[A] = qA$ for any operator A, and here $q = \rho_G$ is the steady-state density matrix. We can separate out $\mathcal{L} = \mathcal{L}_\mathrm{c} + \mathcal{L}_\mathrm{d}$, where $\mathcal{L}_\mathrm{c}[\rho] = -\mathrm{i}[H_S + H_{LS}, \rho]$ contains the coherent part of the evolution, and the remainder $\mathcal{L}_\mathrm{d}$ contains the dissipative part. Then the coherent part of the right-hand side of Eq. (5) acting on an arbitrary density operator ρ gives (remembering that $\mathcal{T}$ and $\mathcal{T}^{-1}$ are antiunitary superoperators)

$$\begin{aligned}
\mathcal{Q}^{-1}\,\mathcal{T}^{-1}\,\mathcal{L}_\mathrm{c}\,\mathcal{T}\,\mathcal{Q}\,[\rho] &= \rho_G^{-1} \cdot \mathcal{T}^{-1}\left[-\mathrm{i}\Big[H_S + H_{LS}, \mathcal{T}[\rho_G \rho]\Big]\right] \\
&= +\mathrm{i}\rho_G^{-1}\Big[\mathcal{T}^{-1}[H_S + H_{LS}], \rho_G \rho\Big] = +\mathrm{i}\Big[H_S + H_{LS}, \rho_G^{-1} \rho_G \rho\Big] = \mathcal{L}_\mathrm{c}^\dagger[\rho],
\end{aligned} \tag{S38}$$

where we have used the fact that that $H_S + H_{LS}$ is TRS-invariant and commutes with ρ_G^{-1}. Now the dissipative part is

$$\begin{aligned}
\mathcal{Q}^{-1}\,\mathcal{T}^{-1}\,\mathcal{L}_{\rm d}\,\mathcal{T}\,\mathcal{Q}\,[\rho] &= \rho_G^{-1}\cdot\mathcal{T}^{-1}\left[\sum_\omega\sum_{\alpha\beta}\tilde{\Gamma}_{\alpha\beta}(\omega)\left[A_\beta(\omega)\rho_G\rho A_\alpha(\omega)^\dagger-\frac{1}{2}\Big\{A_\alpha(\omega)^\dagger A_\beta(\omega),\mathcal{T}[\rho_G\rho]\Big\}\right]\right]\\
&= \rho_G^{-1}\sum_\omega\sum_{\alpha\beta\gamma\delta}[u^{-1}]_{\alpha\gamma}[u^{-1}]_{\beta\delta}\tilde{\Gamma}_{\alpha\beta}(\omega)^*\left(A_\delta(\omega)\rho_G\rho A_\gamma(\omega)^\dagger-\frac{1}{2}\Big\{A_\gamma(\omega)^\dagger A_\delta(\omega),\rho_G\rho\Big\}\right)\\
&= \rho_G^{-1}\sum_\omega\sum_{\gamma\delta}\tilde{\Gamma}_{\gamma\delta}(\omega)\left(A_\delta(\omega)\rho_G\rho A_\gamma(\omega)^\dagger-\frac{1}{2}\Big\{A_\gamma(\omega)^\dagger A_\delta(\omega),\rho_G\rho\Big\}\right)\\
&= \sum_\omega\sum_{\gamma\delta}\tilde{\Gamma}_{\gamma\delta}(\omega)\left(e^{-\beta\omega}A_\delta(\omega)\rho_G\rho A_\gamma(\omega)^\dagger-\frac{1}{2}\Big\{A_\gamma(\omega)^\dagger A_\delta(\omega),\rho_G\rho\Big\}\right)\\
&= \sum_\omega\sum_{\gamma\delta}\tilde{\Gamma}_{\delta\gamma}(-\omega)A_\delta(\omega)\rho_G\rho A_\gamma(\omega)^\dagger-\frac{1}{2}\tilde{\Gamma}_{\gamma\delta}(\omega)\Big\{A_\gamma(\omega)^\dagger A_\delta(\omega),\rho_G\rho\Big\},
\end{aligned}\tag{S39}$$

having used the transformation properties of A_α under $\mathcal{T}^{-1}$ [Eq. (S34) with u replaced by u^{-1}] in the second equality; Eq. (S36) in the third equality; and Eq. (S35) in the final equality. This can be compared to $\mathcal{L}_{\rm d}^\dagger$

$$\begin{aligned}
\mathcal{L}_{\rm d}^\dagger[\rho] &= \sum_\omega\sum_{\gamma\delta}\tilde{\Gamma}_{\gamma\delta}(\omega)^*\left(A_\delta(\omega)^\dagger\rho A_\gamma(\omega)-\frac{1}{2}\Big\{A_\delta(\omega)^\dagger A_\gamma(\omega),\rho\Big\}\right)\\
&= \sum_\omega\sum_{\gamma\delta}\tilde{\Gamma}_{\delta\gamma}(\omega)\left(A_\delta(-\omega)\rho A_\gamma(-\omega)^\dagger-\frac{1}{2}\Big\{A_\delta(\omega)^\dagger A_\gamma(\omega),\rho\Big\}\right),
\end{aligned}\tag{S40}$$

having used $A_\alpha(-\omega)=A_\alpha(\omega)^\dagger$. Direct comparison verifies that (S39) and (S40) are indeed equal, thus confirming that Eq. (5) is satisfied.

4. LINEAR RESPONSE IN OPEN QUANTUM SYSTEMS

This section shows that the retarded Green's function in Eq. (13) in the main text can be directly probed in tunneling spectroscopy experiments (after taking the Fourier transform from the temporal to the frequency domain). We demonstrate that the standard formula for closed systems also applies to open quantum systems upon a generalization of operator time evolution.

Kubo formula. Consider a general Markovian dynamics in the form

$$\frac{d}{dt}\rho=\Big(\mathcal{L}_0+\lambda\mathcal{L}_1(t)\Big)\rho,\tag{S41}$$

where $\mathcal{L}_0$ is the unperturbed Lindbladian, $\lambda\mathcal{L}_1(t)$ is time-dependent perturbation superoperator, and λ is a small perturbation parameter. Assume $\mathcal{L}_0(t)=\mathcal{L}_1\Theta(t-t_0)$ for some initial time t_0, where $\Theta(t-t')$ is the Heaviside step function. We also choose the initial state to be the steady state of the unperturbed system, $\rho(t_0)=\rho_{\rm SS}$ defined as $\mathcal{L}_0\rho_{\rm SS}=0$. In the lowest order of perturbative expansion over λ, the system's dynamics can be represented by a Dyson series,

$$\begin{aligned}
\rho(t) &= \rho_{\rm SS}+\lambda\int_{t_0}^t dt'\exp\left(\mathcal{L}_0(t-t')\right)\mathcal{L}_1\exp(\mathcal{L}_0t')\rho_{\rm SS}+O(\lambda^2)\\
&= \rho_{\rm SS}+\lambda\int_{t_0}^t dt'\exp\left(\mathcal{L}_0(t-t')\right)\mathcal{L}_1\rho_{\rm SS}+O(\lambda^2).
\end{aligned}\tag{S42}$$

Now let us consider the time-dependent expectation value of a local observable O defined as $O(t)=\mathrm{Tr}\,(O\rho(t))$. Its dynamics can be expressed using Eq. (S42) as

$$\begin{aligned} O(t) &= \langle O\rangle_{\rm SS} + \lambda\int_{t_0}^{t} dt'\,{\rm Tr}\left(O\exp\left(\mathcal{L}_0(t-t')\right)\mathcal{L}_1\rho_{\rm SS}\right) + O(\lambda^2) \\ &= \langle O\rangle_{\rm SS} + \lambda\int_{t_0}^{t} dt'\,{\rm Tr}\left(\rho_{\rm SS}\mathcal{L}_1^\dagger\exp\left(\mathcal{L}_0^\dagger(t-t')\right)O\right) + O(\lambda^2) \\ &= \langle O\rangle_{\rm SS} + \lambda\int_{t_0}^{t} dt'\,{\rm Tr}\left(\rho_{\rm SS}\mathcal{L}_1^\dagger O(t-t')\right) + O(\lambda^2), \end{aligned} \tag{S43}$$

where $\langle O\rangle_{\rm SS} = {\rm Tr}\,O\rho_{\rm SS}$, $O(t) = \exp(\mathcal{L}_0^\dagger t)O$, and $\mathcal{L}_0^\dagger, \mathcal{L}_1^\dagger$ are conjugate Liouvillian operators as defined in Eq. (6) in the main text. We focus on unitary perturbation, $\mathcal{L}_1 = -i[H_1,\cdot]$, where H_1 is perturbation Hamiltonian. Extending $t_0 \to -\infty$, we obtain

$$O(t) = \langle O\rangle_{\rm SS} - i\lambda\int_{-\infty}^{\infty} dt'\Theta(t-t')\Big\langle[O(t-t'),H_1]\Big\rangle_{\rm SS} + O(\lambda^2). \tag{S44}$$

For unitary dynamics, this expression coincides with the conventional Kubo formula.

Tunneling spectroscopy. To measure the spectral properties of the system, we assume that the system of interest is connected to a tunneling probe as shown in Fig. 1 in the main text. The probe contains a reservoir of electrons able to tunnel into the system, thus generating an electric current. The dependence of the current on the probe's chemical potential, i.e. differential conductance, can be used to find the spectral function of the system.

Let us consider Eq. (S41) applied to a joint fermionic system-probe configuration. Without the perturbation, we assume that the system and the probe are decoupled, i.e. the unperturbed Liouvillian has the form $\mathcal{L}_0 = \mathcal{L}_0^S + \mathcal{L}_0^R$, where $\mathcal{L}_0^S$ and $\mathcal{L}_0^R$ are Liouville operators acting on system or the probe, respectively. Also, we assume that the initial state is a product state, $\rho_{\rm SS} = \rho_{\rm SS}^S\otimes\rho_{\rm SS}^R$. The coupling between the system and the probe is produced by the perturbation

$$H_1 = \sum_{\mu\nu}\left(T_{\mu\nu}f_\mu^\dagger b_\nu + {\rm h.c.}\right) = B + B^\dagger, \tag{S45}$$

where f_μ and b_ν are Fock operators for the system and the probe respectively, $T_{\mu\nu}$ are tunnelling coefficients, and $B = \sum_{\mu\nu}T_{\mu\nu}f_\mu^\dagger b_\nu$. Here and below μ and ν are generalized indices that incorporate several quantum numbers, including the electron's position and spin.

The tunneling current is defined through the change of the probe's electron number $N_D = \sum_\nu b_\nu^\dagger b_\nu$, namely

$$I := i\lambda[H_1, N_D] = -i\lambda\sum_{\mu\nu}\left(T_{\mu\nu}f_\mu^\dagger b_\nu - {\rm h.c.}\right) = -i\lambda(B - B^\dagger). \tag{S46}$$

Using the Kubo formula in Eq. (S44), we obtain

$$I(t) = 2\lambda^2{\rm Re}\int_{-\infty}^{\infty} dt'\Theta(t-t')\Big(\big\langle\big[B^\dagger(t-t'),B\big]\big\rangle_{\rm SS} - \big\langle\big[B(t-t'),B\big]\big\rangle_{\rm SS}\Big), \tag{S47}$$

where $B(t) = \sum_{\mu\nu}T_{\mu\nu}f_\mu^\dagger(t)b_\nu(t)$ including $f_\mu(t) := \eta_d\exp(\mathcal{L}_0^{S\dagger}t)[\eta_d f_\mu]$ and $b_\nu(t) := \eta_d\exp(\mathcal{L}_0^{R\dagger}t)[\eta_d b_\nu]$, where η_d is dummy Majorana fermion operator as defined in Eq. (S11) (see discussion in Section 1). Assuming that the probe is in the normal (i.e. not superconducting) state, the last term in Eq. (S47) vanishes.

$$\begin{aligned} I(t) &= 2\lambda^2{\rm Re}\int_{-\infty}^{\infty} dt'\Theta(t-t')\sum_{\mu\nu}\sum_{\mu'\nu'}T^*_{\mu\nu}T_{\mu'\nu'}\left(\langle f_\mu^\dagger(t-t')f_{\mu'}\rangle_{\rm SS}\langle b_\nu(t-t')b_{\mu'}^\dagger\rangle_{\rm SS} - \langle f_\mu(t-t')f_{\mu'}^\dagger\rangle_{\rm SS}\langle b_\nu^\dagger(t-t')b_{\mu'}\rangle_{\rm SS}\right) \\ &= 2\lambda^2{\rm Re}\int_{-\infty}^{0} d\tau\sum_{\mu\nu}\sum_{\mu'\nu'}T^*_{\mu\nu}T_{\mu'\nu'}\left(G^<_{\mu\mu'}(\tau)D^>_{\nu\nu'}(-\tau) - G^>_{\mu\mu'}(\tau)D^<_{\nu\nu'}(-\tau)\right) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}I(\omega), \end{aligned} \tag{S48}$$

where $G^<_{\mu\mu'}(\tau) = -i\langle f_\mu^\dagger(\tau)f_{\mu'}\rangle_{\rm SS}$ and $G^>_{\mu\mu'}(\tau) = -i\langle f_\mu^\dagger f_{\mu'}(\tau)\rangle_{\rm SS}$ are the greater and the lesser Green's function, $D^<_{\nu\nu'}(\tau)$ and $D^<_{\nu\nu'}(\tau)$ are similar expressions for the probe in terms of b_ν, and $I(\omega)$ are the Fourier components of the tunneling current defined as

$$I(\omega) = \lambda^2 \sum_{\mu\nu} \sum_{\mu'\nu'} T^*_{\mu\nu} T_{\mu'\nu'} \Big(G^<_{\mu\mu'}(\omega) D^>_{\nu}(\omega) - G^>_{\mu\mu'}(\omega) D^<_{\nu}(\omega) \Big). \tag{S49}$$

Let us assume that the probe's reservoir of fermions is at thermal equilibrium at zero temperature and b_ν are eigenmodes of the reservoir. Then, the expression for probe's Green's functions takes the form

$$D^>_{\nu\nu'}(\omega) = -i\delta_{\nu\nu'} \tilde{A}_\nu(\mu+\omega)\Big(1 - n_F(\mu+\omega)\Big), \qquad D^<_{\nu\nu'}(\omega) = i\delta_{\nu\nu'} \tilde{A}_\nu(\mu+\omega) n_F(\mu+\omega), \tag{S50}$$

where $n_F(x) = \Theta(x)$ is zero-temperature Fermi distribution, μ is probe's chemical potential, and $\tilde{A}_\nu$ is the probe's spectral function.

Assuming $\tilde{A}_\nu(V+\omega) \simeq \tilde{A}_\nu$ depends weakly on the chemical potential V, and introducing coefficients $J_{\mu\mu'} = 2\lambda^2 \sum_\nu T^*_{\mu\nu} T_{\mu'\nu} \tilde{A}_\nu$, we obtain the expression for the differential conductance

$$\frac{\partial I(\omega)}{\partial \mu} = \sum_{\mu\mu'} J_{\mu\mu'} \mathrm{Im}\, G^R_{\mu\mu'}(\omega) \delta(\omega+\mu), \tag{S51}$$

where $\delta(x)$ is the Dirac delta function, $G^R_{\mu\mu'}(\omega) = G^<_{\mu\mu'}(\omega) + G^>_{\mu\mu'}(\omega)$ is the Fourier transform of the time-domain retarded Green's function defined in Eq. (13) in the main text. Choosing appropriate filtering $J_{i\sigma',j\sigma''} \propto \delta_{ij}\delta_{\sigma'\sigma''}\delta_{\sigma\sigma'}$, we can measure spin-polarized current characteristics,

$$\frac{\partial I_\sigma}{\partial \mu} \propto \mathrm{Im}\, G^R_{i\sigma}(-\mu). \tag{S52}$$

Thus, changing the chemical potential μ, we can probe the spectral function of the system.

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