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Exactly soluble model of boundary degeneracy

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We investigate the topological degeneracy that can be realized in Abelian fractional quantum spin Hall states with multiply connected gapped boundaries. Such a topological degeneracy (also dubbed as "boundary degeneracy") does not require superconducting proximity effect and can be created by simply applying a depletion gate to the quantum spin Hall material and using a generic spin-mixing term (e.g., due to backscattering) to gap out the edge modes. We construct an exactly soluble microscopic model manifesting this topological degeneracy and solve it using the recently developed technique [S. Ganeshan and M. Levin, Phys. Rev. B **93**, 075118 (2016)]. The corresponding string operators spanning this degeneracy are explicitly calculated. It is argued that the proposed scheme is experimentally reasonable.

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I. INTRODUCTION

There has been significant recent interest and progress in constructing theoretical models that exhibit exotic, non-Abelian anyons as either intrinsic excitations or states captured by extrinsic defects in various topological phases [1]. Of particular interest here is the possibility to create such non-Abelian anyons in otherwise Abelian topological states. This was explicitly demonstrated in theoretical proposals featuring fractional (Abelian) topological states proximity-coupled to superconductors and in bilayer quantum Hall states with extrinsic twist defects [2–7]. However, there are serious challenges in the experimental realization of these parafermionic models due to a number of poorly compatible ingredients that have to coexist in a single system (in particular, superconductivity and topological order). Moreover, in most cases, braiding properties of the non-Abelian anyons are not sufficiently rich to host universal topological quantum computation.

Recent works have shown that multiple gapped boundaries connected with a common topological bulk can play the role of non-Abelian excitations as long as the bulk supports an intrinsic Abelian topological order [8–12]. The topological ground-state degeneracy in these systems has been dubbed as "boundary degeneracy." In a recent preprint, Barkeshli and Freedman put forward that topological order with a multiply connected gapped boundary can manifest a richer set of topologically protected unitary transformations [13], raising the possibility of realizing universal quantum computation in systems with no superconducting proximity.

The simplest system that is a candidate for manifesting boundary degeneracy is a fractional quantum spin Hall (FQSH) state of filling fraction $\nu = 1/k$ with multiple holes with a boundary (which can be created using a depletion gate) (Fig. 1). Each hole will manifest two counter-propagating edge modes corresponding to the two components of spin. We model these edge modes by chiral Luttinger liquids with opposite chiralities. If we allow direct tunneling between the two edge theories, it would gap them out. Punching out N holes and gluing the two spin components together along the edges is equivalent to creating a fractional quantum Hall state on a manifold of genus $N-1$ [8,12,14], which is known to possess the topological degeneracy k^{N-1}. This proposal for creating topological degeneracy is conceptually simple and could be experimentally implemented immediately when a FQSH is realized. Furthermore, magnetic impurities, which were thought as a nuisance in the current experimental works on QSH effect, can be an advantage toward gapping the edge modes of a FQSH system, which is a necessary step in engineering our topological degeneracy.

In this work, we construct an exactly soluble microscopic model manifesting topological boundary degeneracy. Our construction is rooted in the recently developed [15] Hamiltonian formulation. The relevant topological physics manifests in the effective Hilbert space in the nonperturbative backscattering limit. Within this framework, we prove the existence of a robust topological degeneracy and derive the string operators that span this degeneracy. Our approach in this sense differs from the topological quantum field theory methods [12] and effective boundary action analysis [8]. Toward the end, we outline possible experimental platforms to engineer and probe topological degeneracy via multiply connected gapped boundaries.

II. MODEL

We begin with a microscopic model for a perfectly clean homogeneous edge of the ith hole modeled by two chiral Luttinger liquids with opposite chiralities, one for each spin direction. We then add nonperturbative backscattering terms that mix the two spin components and gap the edge modes. Finally, we connect all the edges (holes) by a common fractionalized bulk. The formalism we consider naturally incorporates this as a constraint on the allowed charge at the edge. We now construct and systematically solve a microscopic model that encapsulates all these aspects.

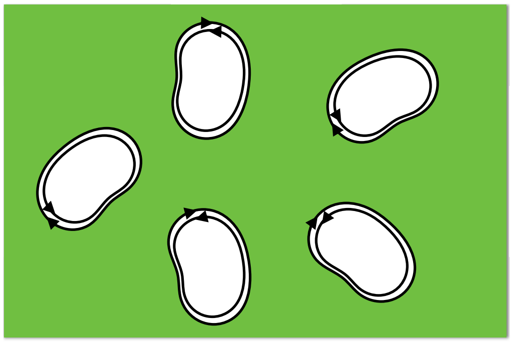

FIG. 1. FQSH phase (green shading) on a multiply connected 2D surface. Interface between holes (in white) manifest two counterpropagating edge modes corresponding to the two spin components.

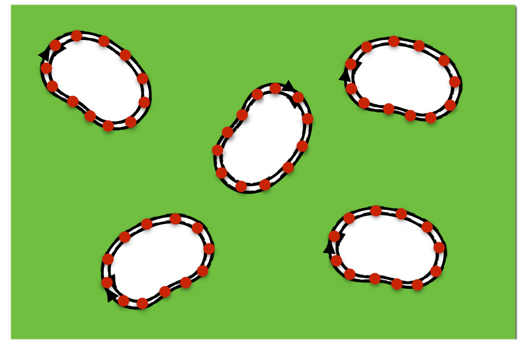

FIG. 2. FQSH phase (green shading) with gapped boundaries. Red dots denote an array of magnetic impurities that gap edges in the limit of continuum backscattering.

The Hamiltonian for a perfectly clean, homogeneous edge of the ith hole is given by

$$H_0^i = \frac{kv_i}{4\pi}\int_{-L/2}^{L/2}\{[\partial_x\phi_\uparrow^i(x)]^2 + [\partial_x\phi_\downarrow^i(x)]^2\}dx, \qquad (1)$$

where v is the velocity of the edge modes of circumference L. $\phi^i_{\uparrow/\downarrow}$ are bosonic fields satisfying canonical commutation relations $[\phi^i_\sigma(x),\partial_y\phi^j_{\sigma'}(y)] = \delta_{ij}\delta_{\sigma\sigma'}\frac{2\pi i}{k_\sigma}\delta(x-y)$, where $k_\uparrow = -k_\downarrow = k$. The density of spin-up electrons at position y at the ith hole is given by $\rho^i_\uparrow(y) = \frac{1}{2\pi}\partial_y\phi^i_\uparrow$, while the density of spin-down electron is $\rho^i_\downarrow(y) = \frac{1}{2\pi}\partial_y\phi^i_\downarrow$. The total charge Q^i and total spin S^i_z on the edge of the ith hole are given by $Q^i = Q^i_\uparrow + Q^i_\downarrow$ and $S^i_z = \frac{1}{2}(Q^i_\uparrow - Q^i_\downarrow)$, with

$$Q^i_\sigma = \frac{1}{2\pi}\int_{-L/2}^{L/2}\partial_y\phi^i_\sigma dy, \quad \sigma = \uparrow,\downarrow.$$

The spin-up and spin-down electron creation operators at each hole take the form $\psi_\uparrow^{i\dagger} = e^{ik\phi^i_\uparrow}, \quad \psi_\downarrow^{i\dagger} = e^{-ik\phi^i_\downarrow}$. Note that H^i corresponds to a collection of decoupled edge modes, and the key information that these modes are actually multiply connected via a common fractionalized bulk is missing. This multiple connectedness of the holes results in two quantization conditions on $Q^i_\uparrow, Q^i_\downarrow$:

$$Q^i_{\uparrow,\downarrow} \in \mathbb{Z}\times 1/k \quad \text{and} \quad \sum_{i=1}^N Q^i_{\uparrow,\downarrow} \in \mathbb{Z}.$$

Physically, these quantization conditions require that the edge modes corresponding to holes contain fractional charges in multiples of $1/k$ and that the net charge on all the holes adds up to be an integer multiple of the electronic charge. For example, the edge of an isolated single hole cannot carry any excess fractional charge. A closely related fact to this quantization is that the bosonic operators $\phi^i_\uparrow(y)$ and $\phi^i_\downarrow(y)$ are actually *compact* degrees of freedom, which are only defined modulo $2\pi/k$. Following Ref. [15], we dynamically impose the quantization on $Q^i_\uparrow, Q^i_\downarrow$. To this end, we add $H^i_{\rm lq} = -U\cos(2\pi k Q^i_\uparrow) - U\cos(2\pi k Q^i_\downarrow)$ to the edge Hamiltonian of the ith hole. We then impose the second condition, corresponding to the global quantization of the total charge on all holes, by adding a global term $H_{\rm gq} = -U\cos(2\pi\sum_i^N Q^i_\uparrow) - U\cos(2\pi\sum_i^N Q^i_\downarrow)$. Notice that both quantization conditions are imposed by letting $U\to\infty$. The Hilbert space corresponding to the clean edge is spanned by the complete orthonormal basis $\{|q^i_\uparrow, q^i_\downarrow, \{n^i_{p\uparrow}\}, \{n^i_{p\downarrow}\}\rangle\}$, where the quantum numbers $q^i_\uparrow, q^i_\downarrow$ correspond to the total charge associated with the two spin species ranging over $\mathbb{Z}\times 1/k$ (subject to $\sum_i q^i_{\uparrow,\downarrow}\in\mathbb{Z}$), while $n^i_{p\uparrow}, n^i_{p\downarrow}$ are the neutral phonon excitations of momentum p ranging over all nonnegative integers for each value of $p = 2\pi/L, 4\pi/L, \ldots$.

The next step is to add backscattering terms that gap the above defined boundary modes by scattering spin-up electrons to spin-down electrons. A continuum of backscattering terms in a fermionic representation can be expressed as $H^i_{\rm bs} = \int_0^L \frac{U(x)}{2}\psi^{i\dagger}(x)\psi^i(x) + \text{H.c.}$ The corresponding bosonized representation can be written as $H^i_{\rm bs} = \int_0^L U(x)\cos\{k[\phi^i_\uparrow(x) + \phi^i_\downarrow(x)]\}$. The total Hamiltonian for the ith hole $H^i_0 + H^i_{\rm bs}$ corresponds to a gapped edge in the large U limit.

Now we are set to write down the full microscopic Hamiltonian corresponding to the N multiply connected hole boundaries: $H = H_{\rm gq} + \sum_{i=1}^N H^i_0 + H^i_{\rm bs} + H^i_{\rm lq}$. The Hamiltonian H can be mapped onto a class of exactly soluble Hamiltonians by replacing the continuum backscattering term $\int_0^L U(x)\cos\{k[\phi^i_\uparrow(x)+\phi^i_\downarrow(x)]\}$ with an array of M impurity scatterers $U\sum_{j=1}^M \cos\{k[\phi^i_\uparrow(x_j)+\phi^i_\downarrow(x_j)]\}$ (see Fig. 2). The continuum result is then recovered in the thermodynamic limit of $L,M\to\infty$ with U and L/M fixed. Without loss of generality, we periodically arrange the backscattering terms at each hole as $x_{1\ldots M} = 0,\ldots,(M-1)s$, where s is the spacing between two impurity points. After this replacement, the Hamiltonian H is exactly soluble in the limit $U\to\infty$. To make contact with with the formalism outlined in Ref. [15], we rewrite the above model as

$$H = H_0 - U\sum_{i=1}^{N(M+2)+2}\cos(C_i). \qquad (2)$$

In the above notation, the first term $H_0 = \sum_i^N H^i_0$ contains the dynamics of the clean edge. The second term contains the backscattering terms on all the holes and their corresponding charge quantization conditions. We have organized the cosine arguments in the following way. The first NM terms consist of all the backscattering terms $\{C_{1..M},..,C_{(N-1)M+1...NM}\} = \{k[\phi^1_\uparrow(x_{1..M})+\phi^i_\downarrow(x_{1..M})]\dots k[\phi^N_\uparrow(x_{1..M})+\phi^N_\downarrow(x_{1..M})]\}$. The quantization condition of each hole boundary is given by $\{C_{NM+1},..,C_{NM+N}\} = \{2\pi k Q^1_\uparrow \dots 2\pi k Q^N_\uparrow\}$ and $\{C_{NM+N+1},..,C_{NM+2N}\} = \{2\pi k Q^1_\downarrow \dots 2\pi k Q^N_\downarrow\}$. Finally, the two conditions on the total charge are given by $\{C_{N(M+2)+1},C_{N(M+2)+2}\} = \{2\pi \sum_i^N Q^i_\uparrow, 2\pi \sum_i^N Q^i_\downarrow\}$.

III. REVIEW OF FORMALISM

In this section we review the general formalism for solving the class of Hamiltonians central to our discussion,

$$H = H_0 - U\sum_{i=1}^{M}\cos(C_i). \tag{3}$$

Here we have defined H_0 as a quadratic function of position and momentum variables $\{x_1,p_1,x_2,p_2,\dots\}$ and the C_i are linear functions of these variables. We restrict to the case where $\{C_1,C_2,\dots\}$ are linearly independent, $[C_i,C_j]$ is an integer multiple of $2\pi i$ for all i,j such that the cosine terms commute. The detailed recipe for developing a low-energy theory in the large U limit is outlined in Ref. [15]. Here we provide a skeletal recap of this recipe.

In the limit $U \to \infty$, the arguments of the cosine terms are pinned to integer multiples of 2π. The low-energy spectrum of H in this limit can be described by an effective quadratic Hamiltonian H_{eff} acting within an effective Hilbert space $\mathcal{H}_{\text{eff}}$. The effective Hamiltonian is given by

$$H_{\text{eff}} = H_0 - \sum_{i,j=1}^{M}\frac{(\mathcal{M}^{-1})_{ij}}{2}\Pi_i\Pi_j, \tag{4}$$

where the operators $\Pi_1,\dots,\Pi_M$ are defined by $\Pi_i = \frac{1}{2\pi i}\sum_{j=1}^{M}\mathcal{M}_{ij}[C_j,H_0]$ and where $\mathcal{M}_{ij}$ is a matrix defined by $\mathcal{M} = \mathcal{N}^{-1}$, $\mathcal{N}_{ij} = -\frac{1}{4\pi^2}[C_i,[C_j,H_0]]$. Π_i operators satisfy $[C_i,\Pi_j] = 2\pi i\delta_{ij}$ by construction.

The simple physical intuition is that the low-energy physics of H in the limit $U \to \infty$ does not contain the dynamics of C_i's. Thus, the term generating the dynamics $\frac{(\mathcal{M}^{-1})_{ij}}{2}\Pi_i\Pi_j$ must be removed from the effective Hamiltonian.

This effective Hamiltonian is defined on an effective Hilbert space $\mathcal{H}_{\text{eff}}$, which is a *subspace* of the original Hilbert space $\mathcal{H}$ and which consists of all states $|\psi\rangle$ satisfying

$$\cos(C_i)|\psi\rangle = |\psi\rangle, \quad i = 1,\dots,M. \tag{5}$$

We can directly find the creation and annihilation operators for H_{eff} by finding all operators a that obey $[a,H_{\text{eff}}] = Ea$. Finding creation and annihilation operators for the H_{eff} is equivalent to solving the equation

$$[a,H_0] = Ea + \sum_j \lambda_j[C_j,H_0], \quad [a,C_i] = 0, \quad i = 1,2,\dots \tag{6}$$

where λ_j'a are like Lagrange multipliers imposing the constraint due to the large cosine terms and E is arbitrary scalar with $E \neq 0$ ($E > 0$ corresponds to "annihilation operators," and $E < 0$ corresponds to "creation operators." The normalized creation and annihilation operators satisfy

$$[a_k,a^\dagger_{k'}] = \delta_{kk'}, \quad [a_k,a_{k'}] = [a^\dagger_k,a^\dagger_{k'}] = 0. \tag{7}$$

We now construct a complete set of quantum numbers for labeling the eigenstates of H_{eff}. This step nicely fleshes out the physical structure of the effective Hilbert space $\mathcal{H}_{\text{eff}}$. With this motivation in mind, consider the object $\mathcal{Z}_{ij}$ to be the $M \times M$ matrix defined by

$$\mathcal{Z}_{ij} = \frac{1}{2\pi i}[C_i,C_j]. \tag{8}$$

The matrix $\mathcal{Z}_{ij}$ is integer and skew-symmetric, but otherwise arbitrary. Next, let

$$C'_i = \sum_{j=1}^{M}\mathcal{V}_{ij}C_j + \chi_i \tag{9}$$

for some matrix $\mathcal{V}$ and some vector χ. Then, $[C'_i,C'_j] = 2\pi i\mathcal{Z}'_{ij}$ where $\mathcal{Z}' = \mathcal{V}\mathcal{Z}\mathcal{V}^T$. The second step of the recipe is to find a matrix $\mathcal{V}$ with integer entries and determinant ±1, such that $\mathcal{Z}'$ takes the simple form

$$\mathcal{Z}' = \begin{pmatrix} 0_I & -\mathcal{D} & 0 \\ \mathcal{D} & 0_I & 0 \\ 0 & 0 & 0_{M-2I}\end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_I\end{pmatrix}. \tag{10}$$

Here I is some integer with $0 \leqslant I \leqslant M/2$ and 0_I denotes an $I \times I$ matrix of zeros. In mathematical language, $\mathcal{V}$ is an integer change of basis that puts $\mathcal{Z}$ into *skew-normal* form. It is known that such a change of basis always exists, though it is not unique. After finding an appropriate $\mathcal{V}$, the offset χ should then be chosen so that

$$\chi_i = \pi\sum_{j<k}\mathcal{V}_{ij}\mathcal{V}_{ik}\mathcal{Z}_{jk} \pmod{2\pi}. \tag{11}$$

The reason for choosing χ in this way is that it ensures that $e^{iC'_i}|\psi\rangle = |\psi\rangle$ for any $|\psi\rangle \in \mathcal{H}_{\text{eff}}$, as can be easily seen from the Campbell-Baker-Hausdorff formula.

The complete low-energy spectrum of H_{eff} can always be written in the form

$$H_{\text{eff}} = \sum_{k=1}^{K}E_k a^\dagger_k a_k + F(C'_{2I+1},\dots,C'_M), \tag{12}$$

where F is some (*a priori* unknown) quadratic function. As a consequence of this construction, we note that the following operators all commute with each other:

$$\{e^{iC'_1/d_1},\dots,e^{iC'_I/d_I}, e^{iC'_{I+1}},\dots,e^{iC'_{2I}}, C'_{2I+1},\dots,C'_M, a^\dagger_{1..K}a_{1..K}\}. \tag{13}$$

We denote the simultaneous eigenstates by

$$|\theta_1,\dots,\theta_I,\ \varphi_1,\dots,\varphi_I,\ x'_{I+1},\dots,x'_{M-I},n_1,\dots,n_K\rangle,$$

or, in more abbreviated form, $|\boldsymbol{\theta},\boldsymbol{\varphi},\boldsymbol{x}',\boldsymbol{n}\rangle$.

By construction, the $|\boldsymbol{\theta},\boldsymbol{\varphi},\boldsymbol{x}',\boldsymbol{n}\rangle$ states form a complete basis for the Hilbert space $\mathcal{H}$. A *subset* of these states form a complete basis for the effective Hilbert space $\mathcal{H}_{\text{eff}}$. This subset consists of all $|\boldsymbol{\theta},\boldsymbol{\varphi},\boldsymbol{x}',\boldsymbol{n}\rangle$ for which

(1) $\boldsymbol{\theta} = (2\pi\alpha_1/d_1,\dots,2\pi\alpha_I/d_I)$ with α_i=0,1,...,d_i-1.

(2) $\boldsymbol{\varphi} = (0,0,\dots,0)$.

(3) $(x'_{I+1},\dots,x'_{M-I}) = (q_1,\dots,q_{M-2I})$ for some integers q_i.

We will denote this subset of eigenstates by $\{|\boldsymbol{\alpha},\boldsymbol{q},\boldsymbol{n}\rangle\}$. Putting this together, we can see that the $|\boldsymbol{\alpha},\boldsymbol{q},\boldsymbol{n}\rangle$ are eigenstates of H_{eff}, with eigenvalues

$$E = \sum_{k=1}^{K} n_k E_k + F(2\pi q_1,\dots,2\pi q_{M-2I}). \tag{14}$$

A key feature of Eq. (14) that is worth mentioning is that the energy E is independent of the quantum numbers $\alpha_1,\dots,\alpha_I$. Since α_i ranges from $0 \leqslant \alpha_i < d_i - 1$, it follows that every eigenvalue of H_{eff} has a degeneracy of (at least)

$$D = \prod_{i=1}^{I} d_i. \tag{15}$$

IV. FULL-ENERGY SPECTRUM WITH BOUNDARY DEGENERACY

In this section we apply the recipe outlined in the previous section to the Hamiltonian defined in Eq. (2). W calculate the low-energy effective Hamiltonian H_{eff} and the low-energy Hilbert space $\mathcal{H}_{\text{eff}}$ corresponding to H in the limit $U \to \infty$. The effective Hamiltonian is given by

$$H_{\text{eff}} = H_0 - \sum_{i,j=1}^{N(M+2)+2} \frac{(\mathcal{M}^{-1})_{ij}}{2} \Pi_i \Pi_j, \tag{16}$$

where the operators $\Pi_1,\dots,\Pi_{N(M+2)+2}$ are defined by $\Pi_i = \frac{1}{2\pi i}\sum_{j=1}^{N(M+2)+2} \mathcal{M}_{ij}[C_j,H_0]$ and where $\mathcal{M}_{ij}$ is a matrix defined by $\mathcal{M} = \mathcal{N}^{-1}$, $\mathcal{N}_{ij} = -\frac{1}{4\pi^2}[C_i,[C_j,H_0]]$. Π_i operators satisfy $[C_i,\Pi_j] = 2\pi i \delta_{ij}$ by construction. This effective Hamiltonian is defined on an effective Hilbert space $\mathcal{H}_{\text{eff}}$, which is a *subspace* of the original Hilbert space $\mathcal{H}$ and which consists of all states $|\psi\rangle$ satisfying $\cos(C_i)|\psi\rangle = |\psi\rangle$, $i = 1,\dots,N(M+2)+2$. We can directly find the creation and annihilation operators for H_{eff} by finding all operators a that obey $[a,H_{\text{eff}}] = Ea$. Putting this all together, we see that the most general possible creation and annihilation operator for H_{eff} is given by

$$a_{ipm} = \sqrt{\frac{k}{4\pi|p|s}} \int_{-L/2}^{L/2} [(e^{ipy}\partial_y\phi^i_\uparrow + e^{2ipx_m}e^{-ipy}\partial_y\phi^i_\downarrow) \times \Theta(x_{m-1} < y < x_m)]dy.$$

Here the index m runs over $m = 1,\dots,M$, i runs over the holes $i = 1,\dots,N$, while p takes values $\pm\pi/s, \pm 2\pi/s,\dots$. The operators are normalized to yield $[a_{ipm},a^\dagger_{i'p'm'}] = \delta_{pp'}\delta_{mm'}\delta_{ii'}$ for $p,p' > 0$. The cosine terms imposing quantization and compactness condition naturally forbids a to be an explicit function of the bosonic field ϕ^i.

We now construct a complete set of commuting operators for labeling the eigenstates of H_{eff}. In order to do this, we consider the integer and skew-symmetric $[N(M+2)+2] \times [N(M+2)+2]$ matrix $\mathcal{Z}_{ij}$ defined by $\mathcal{Z}_{ij} = \frac{1}{2\pi i}[C_i,C_j]$. Let $C'_i = \sum_{j=1}^{N(M+2)+2} \mathcal{V}_{ij}C_j + \chi_i$ for some matrix $\mathcal{V}$ such that $[C'_i,C'_j] = 2\pi i \mathcal{Z}'_{ij}$, where $\mathcal{Z}' = \mathcal{V}\mathcal{Z}\mathcal{V}^T$. The offset χ_i must be chosen to be $\chi_i = \pi \cdot \sum_{j<k} \mathcal{V}_{ij}\mathcal{V}_{ik}\mathcal{Z}_{jk} \pmod{2\pi}$ such that $e^{iC'_i}|\psi\rangle = |\psi\rangle$ is satisfied for any $|\psi\rangle \in \mathcal{H}_{\text{eff}}$. We then find a matrix $\mathcal{V}$ with integer entries and determinant ± 1, such that $\mathcal{Z}'$ takes the simple form

$$\mathcal{Z}' = \begin{pmatrix} 0_N & -\mathcal{D}_N & 0 \\ \mathcal{D}_N & 0_N & 0 \\ 0 & 0 & 0_{NM+2} \end{pmatrix}, \quad \mathcal{D}_N = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & k \end{pmatrix}. \tag{17}$$

Here 0_N denotes an $N \times N$ matrix of zeros. $\mathcal{V}$ is an integer change of basis that puts $\mathcal{Z}$ into *skew-normal* form. In the $\boldsymbol{C}'$ basis, the diagonalized low-energy effective Hamiltonian H_{eff} takes the form

$$H_{\text{eff}} = \sum_{i=1}^{N}\sum_{m=1}^{M}\sum_{p>0} vpa^\dagger_{ipm}a_{ipm} + F(C'_{2N+1},\dots,C'_{N(M+2)+2}), \tag{18}$$

where the sum runs over $p = \pi/s, 2\pi/s,\dots$ and where F is some quadratic function of $NM+2$ variables associated with the 0_{NM+2} block of the $\mathcal{Z}'_{ij}$ matrix. The exact form of F does not play a role in the analysis to follow and we keep it general (even though it can be computed following Ref. [15]). Using the commutation algebra of the C'_i operators, we can construct the complete set of operators that commute with each other and with H_{eff}. The effective Hilbert space $\mathcal{H}_{\text{eff}}$ is then spanned by the unique simultaneous eigenstates $\{|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle\}$, satisfying

$$\begin{aligned} e^{iC'_{1,N+1}}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle &= |\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle, \\ e^{iC'_{2,\dots,N}/k}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle &= e^{i2\pi\alpha_{2..N}/k}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle, \\ e^{iC'_{N+2,\dots,2N}}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle &= |\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle, \\ C'_{2N+1\dots N(M+2)+2}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle &= 2\pi q_{1..NM+2}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle, \\ a^\dagger_{ipm}a_{ipm}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle &= n_{ipm}|\boldsymbol{\alpha},\boldsymbol{q},\{n_{ipm}\}\rangle. \end{aligned} \tag{19}$$

Here the label n_{ipm} runs over nonnegative integers, while $\boldsymbol{\alpha}$ is an abbreviation for the $(N-1)$-component integer vector $(\alpha_2,\dots,\alpha_N)$ where $\alpha_{2..N}$'s run over $\{0\dots k-1\}$. $\{|\boldsymbol{\alpha},\boldsymbol{q},\{n_{pm}\}\rangle\}$ basis states are also eigenstates of H_{eff} with the total energy given by

$$E = \sum_{i=1}^{N}\sum_{m=1}^{M}\sum_{p>0} vpn_{ipm} + F(2\pi q_1,\dots,2\pi q_{NM+2}). \tag{20}$$

There are two important features of the above spectrum. a) E has a finite *energy gap* of order v/s where $s = L/M$. b) The spectrum E is independent of the quantum numbers $\boldsymbol{\alpha}$. In other words, every state, including the ground state, has a degeneracy of

$$D = k^{N-1}, \tag{21}$$

since this is the number of different values that $\boldsymbol{\alpha}$ ranges over. This degeneracy agrees with the prediction made in the introduction. The physical origin of this degeneracy can be traced to the equivalence of our system to that of FQH state with filling factor $1/k$ on a $N-1$ genus manifold (see schematic Fig. 3).

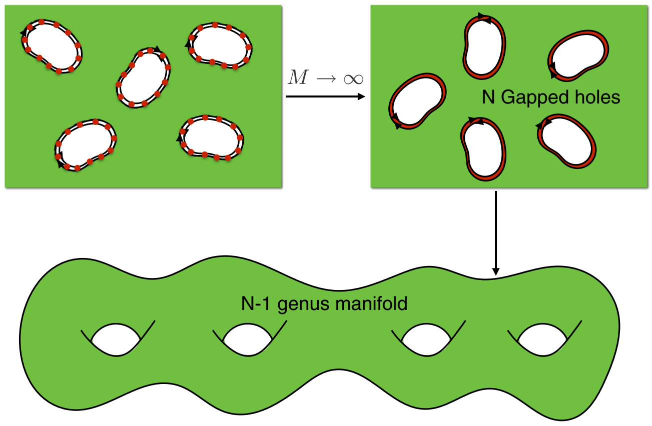

FIG. 3. FQSH phase (green shading) with gapped boundaries. Red dots denote an array of magnetic impurities that gap edges in the limit of continuum backscattering (right). N gapped boundaries multiply connected through a bulk is equivalent to a FQH state of $1/k$ filling on a $N-1$ genus manifold.

V. STRING OPERATORS

In the above analysis, we were able to identify quantum numbers and the complete set of commuting operators associated with the effective Hilbert space. From these commuting operators we can deduce the so-called "string operators" that span the degenerate subspace. The string operators in the primed basis are given by $\{e^{iC'_{2,\dots,N}/k}, e^{iC'_{N+2,\dots,2N}/k}\}$. In the unprimed basis, these operators are defined as

$$\{e^{i2\pi(Q^i_\uparrow - Q^i_\downarrow)}, \prod_{r=1}^{j} e^{i(\phi^r_\uparrow(x)+\phi^r_\downarrow(x))}\},$$
$$i = 1,\dots,N-1,\; j = i+1,\dots. \tag{22}$$

Note that the above operators are closely related to the parafermion operators and are fixed by the nonunique choice of $\mathcal{V}$. One can obtain the matrix representation of these string operators by acting in the basis states spanned by the degenerate ground-state subspace $|\boldsymbol{\alpha},0,0\rangle \equiv |\boldsymbol{\alpha}\rangle$:

$$e^{\pm iC'_i/k}|\boldsymbol{\alpha}\rangle = e^{\pm i2\pi\alpha_i/k}|\boldsymbol{\alpha}\rangle,$$
$$e^{\pm iC'_{i+N}/k}|\boldsymbol{\alpha}\rangle = |\boldsymbol{\alpha} \pm \boldsymbol{e_{i-1}}\rangle,\; i = 2\dots N.$$

Here $\boldsymbol{e_i}$ denotes the $(N-1)$-component vector $\boldsymbol{e_i} = (0,\dots,1,\dots,0)$ with a "1" in the ith entry and 0 everywhere else. The addition of $\boldsymbol{e_i}$ is performed modulo k. Note that the above equations imply that the operators $e^{\pm iC'_i/k}$ act like "clock" matrices for $i = 2,\dots,N$, while the operators $e^{\pm iC'_i/k}$ act like "shift" matrices for $i = N+2,\dots,2N$; thus, these operators generate a generalized Pauli algebra (a.k.a. σ_z,σ_x).

VI. TOPOLOGICAL ROBUSTNESS

Having established the ground-state degeneracy in the $U \to \infty$ case of our toy model, we proceed to describe finite-U corrections to H_{eff}. Notice that we only seek finite-U corrections to the backscattering terms that gap the edge. In other words, consider Eq. (2) to be of the form $H = H_0 - U\sum_{i=1}^{NM}\cos(C_i) - U'\sum_{i=NM+1}^{N(M+2)+2}\cos(C_i)$ in the limit where U is finite but $U' \to \infty$ (U' are associated with the quantization condition). In this case, the finite-U corrections only generate (instanton-like) tunneling processes of the form $C_i \to C_i - 2\pi n_i$ (for $i = 1\dots NM$).

The thermodynamic limit we consider is where $L,M \to \infty$ with U and L/M fixed. Notice that the boundary corresponding to each hole has a finite *energy gap* in this limit (of order v/s, where $s = L/M$). Due to the gapped spectrum, we can employ perturbative methods to probe the degeneracy. The most general low-energy operator generating finite-U corrections to the ground state can be written as $e^{i\sum_{j=1}^{NM} m_j\Pi_j}\cdot\epsilon_{\boldsymbol{m}}$ with the sum running over the NM-component integer vectors $\boldsymbol{m} = (m_1,\dots,m_{NM})$ [15]. Here, the $\epsilon_{\boldsymbol{m}}$ are unknown functions of $\{a_{ipm},a^\dagger_{ipm},C'_{2N+1,\dots N(M+2)+2}\}$ that vanish in the limit $U \to \infty$. The Π_i operators are conjugate to the C_i's ($[C_i,\Pi_j] = 2\pi i\delta_{ij}$) and thereby generate tunneling events associated with the finite-U corrections. Since the spectrum is gapped in the limit of interest, the ground-state degeneracy and the gap are robust against small perturbations. The lowest-order nonvanishing matrix elements splitting the degeneracy within the ground state come from the simultaneous single-instanton tunneling event at all M impurity points of a given hole ($m_1 = \dots = m_M = 1$, which is an Mth-order instanton process). This lowest-order splitting is suppressed by a factor of $\sim e^{-\text{const.}M\sqrt{U}}$ [15,16], which vanishes in the thermodynamic limit of $M \to \infty$, exemplifying the topological nature of the degeneracy.

VII. EXPERIMENTAL REALIZATION

The proposed model for topological degeneracy can be realized in a variety of systems where edges around punctures of a conjugate pair of Abelian fraction quantum Hall states can be gapped via backscattering. First, an electron-hole bilayer can exhibit the desired pair of conjugate Abelian fractional quantum Hall states, while top and bottom gates can be used to puncture holes, whose edges can be coupled via electron tunneling [14]. Second, a back gate in an electronic FQSH system can be used to puncture holes, while magnetic impurities can be used to flip the spin and thus couple the edges. In Fig. 4, we outline a generalization of an architecture that has been used in fractional quantum Hall experiments [17]. The idea is to create a central depletion region using a back gate. The side gates create a quantum point contact that can

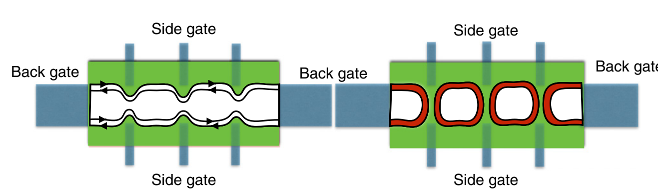

FIG. 4. Schematic to create topological degeneracy (top view): (Left) FQSH (green shading) with an elongated depletion region (white region) controlled by a back gate. The side gates create QPC that weakly scatters electrons across the trench. (Right) FQSH with doped magnetic impurities that gap the edge. Each hole is shown in white with a red shading denoting a gapped boundary. The side gate voltage is tuned to the strong backscattering limit or quasiparticle tunneling regime. The side gates allow exchange of quasiparticles between the disconnected gapped boundaries.

pinch off the trench and create multiply connected regions in the topological state. Notice that in the dual limit after the pinch-off the holes exchange fractional quasiparticles, thereby changing the topological sectors controlled by the side gate.

Third, ultracold dipoles, such as magnetic atoms [18,19], polar molecules [20,21], and Rydberg atoms [22,23], pinned in optical lattices can be used to realize spin models whose ground states correspond to bilayer fractional quantum Hall states [24]. It is possible that the ground state of such a bilayer system can be tuned to the desired conjugate pair of Abelian fractional quantum Hall states, in which case focused laser beams can be used to locally modify the spin model to effectively puncture holes and couple the resulting edges. Fourth, with the help of synthetic gauge fields and contact interactions, two internal states of ultracold atoms can exhibit the FQSH effect, while focused laser beams can be used to puncture holes and induce transitions between the two internal states, thus coupling the edges [25]. Finally, photonic implementations in radio-frequency [26], microwave [27], and optical [28–33] domains can also be envisioned.

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