Supplemental Material for: Dynamical phase transitions in sampling complexity

Abhinav Deshpande,[1,2] Bill Fefferman,[1,3] Minh C. Tran,[1,2] Michael Foss-Feig,[4,1,2] and Alexey V. Gorshkov[1,2]

[1] *Joint Center for Quantum Information and Computer Science, NIST/University of Maryland, College Park, MD 20742, USA*

[2] *Joint Quantum Institute, NIST/University of Maryland, College Park, MD 20742, USA*

[3] *Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA*

[4] *United States Army Research Laboratory, Adelphi, MD 20783, USA*

In this supplemental material, we give expressions for the output probabilities in the distribution $\mathcal{D}_U$ in a boson sampling experiment. We then explicitly present the algorithm and derive the expression for $\mathcal{D}_{DP}$. We then derive an upper bound to the variation distance between them, proving lemma 3 of the main text.

Expression for output probabilities.— In this section, we describe the standard boson sampling set-up and derive an expression for the output probabilities of a boson sampling experiment that define the distribution $\mathcal{D}_U$. First, let us represent the input and output states pictorially and develop some notation.

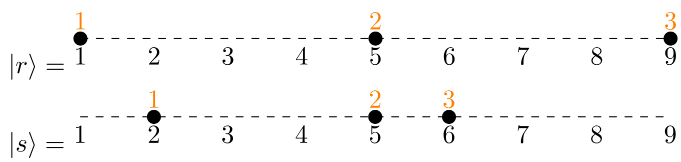

FIG. 1. A representation of input and output basis states in 1-D.

In Fig. 1, the top line denotes the input state $|r\rangle$ and the bottom line the output $|s\rangle$. Each filled circle denotes a boson occupying the corresponding lattice site, which is labeled below the circles. The numbers marked in orange above each boson label the bosons from left to right (more generally, this is in a nondecreasing order of the site index). We will call this the boson index.

A given configuration (basis state) is completely specified by specifying the boson number in each site, such as $r = (1, 0, 0, 0, 1, 0, 0, 0, 1)$ and $s = (0, 1, 0, 0, 1, 1, 0, 0, 0)$ in the above. It can also be specified by listing the site index for every boson index, i.e. the occupied sites. Thus the input state can be represented as $\mathsf{in} = (1, 5, 9)$, the output state as $\mathsf{out} = (2, 5, 6)$.

All $n!$ permutations of the boson indices represent valid paths that the bosons can take to the output state, and correspond to the $n!$ terms in the permanent of the matrix. In cases where there are two or more bosons in a particular site at the input or output, there are $\frac{n!}{r!s!}$ paths (and terms in the amplitude). Here, $r! := r_1!r_2! \ldots r_m!$ and similarly $s!$. By taking repeated rows and columns of R, this has the effect of still giving $n!$ terms in total, which we identify with the $n!$ permutations in the boson indices. The expression for the probability of an outcome s is (here, $b_i := a_i(t)$):

$$\Pr_{\mathcal{D}_U}[s] = \frac{1}{r_1!r_2! \ldots r_m!s_1!s_2! \ldots s_m!} |\langle \mathrm{vac}| b_1^{s_1} b_2^{s_2} \ldots b_m^{s_m} a_1^{\dagger r_1} a_2^{\dagger r_2} \ldots a_m^{\dagger r_m} |\mathrm{vac}\rangle|^2 \tag{S1}$$

$$= \frac{1}{r!s!} |\langle \mathrm{vac}| (\hat{U}^\dagger a_1^{s_1} \hat{U})(\hat{U}^\dagger a_2^{s_2} \hat{U}) \ldots (\hat{U}^\dagger a_m^{s_m} \hat{U}) a_1^{\dagger r_1} a_2^{\dagger r_2} \ldots a_m^{\dagger r_m} |\mathrm{vac}\rangle|^2 \tag{S2}$$

$$= \frac{1}{r!s!} |\langle \mathrm{vac}| \left(\sum_{k_1=1}^{m} R^\dagger_{1k_1} a_{k_1} \right)^{s_1} \ldots \left(\sum_{k_m=1}^{m} R^\dagger_{mk_m} a_{k_m} \right)^{s_m} a_1^{\dagger r_1} a_2^{\dagger r_2} \ldots a_m^{\dagger r_m} |\mathrm{vac}\rangle|^2, \tag{S3}$$

where R_{ij} describes the action of $\hat{U}$ on the annihilation operators at a site: $b_i = a_i(t) = \sum_k R^\dagger_{ik}(t) a_k(0)$. Now define the matrix $A^\dagger$ to be the one obtained by taking s_i copies of the i'th row and r_j copies of the j'th column of $R^\dagger$. For concreteness, this can be done by first considering the rows and repeating a row i of $R^\dagger$ whenever $s_i > 1$, or not picking it if $s_i = 0$, to convert it into an $n \times m$ matrix. We can then do the same with columns to convert it into an $n \times n$ matrix. However, the order ultimately does not matter since the quantity that emerges, the permanent, is symmetric under exchange of rows or columns. We have

$$\Pr_{\mathcal{D}_U}[s] = \frac{1}{r!s!} \left| \sum_\sigma \prod_i R^\dagger_{\mathsf{out}_{\sigma(i)},\mathsf{in}_i} \right|^2 \tag{S4}$$

$$= \frac{1}{r!s!} \left| \sum_\sigma \prod_i A^\dagger_{\sigma(i),i} \right|^2 , \tag{S5}$$

where the sum is over all permutations σ. This finally gives us

$$\Pr_{\mathcal{D}_U}[s] = \frac{1}{r!s!} \left| \mathrm{Per}(A^\dagger) \right|^2 = \frac{1}{r!s!} |\mathrm{Per}(A)|^2, \tag{S6}$$

where $\mathrm{Per}(A)$ is the *permanent* of A.

Algorithm.— The sampling algorithm is given below. It is easy to see that it implements one step of a Markov process of n distinguishable bosons walking on a lattice.

Algorithm 1: Sampling algorithm

Input: Unitary $R(t)$, tolerance ϵ
Output: Sample s drawn from $\mathcal{D}_{DP}$, a distribution that is close to $\mathcal{D}_U$.

1 $\mathcal{P}_{kl} = |R(t)|^2_{kl}$
2 **for** $i \in \{1, 2, \dots n\}$, **do**
3 Select site l from the distribution $\mathcal{P}_{\mathsf{in}_i,l}$ for the boson at $\mathsf{in_i}$ to hop to.
4 Increment output boson number of site l by 1: $s_l \to s_l + 1$ (or equivalently, assign $\mathsf{out}_i = l$)
5 **end**
6 **return** configuration s (or out), a sample from $\mathcal{D}_{DP}$.

Note that $\mathcal{P}$ from line 1 is a doubly stochastic matrix describing the classical Markov process. To see that the runtime is polynomial in n, note that the loop is over n boson indices. Line 3 takes time $O(m \log m) = \tilde{O}(n^\beta)$, giving a total runtime of $\tilde{O}(nm) = \tilde{O}(n^{1+\beta})$. The notation $\tilde{O}$ suppresses factors of $\log n$.

Bound on variation distance.— Here we derive a bound on the variation distance $\|\mathcal{D}_U - \mathcal{D}_{DP}\| = \frac{1}{2}\sum_s |\Pr_{\mathcal{D}_U}(s) - \Pr_{\mathcal{D}_{DP}}(s)|$. Rewriting the actual probability in terms of the amplitudes, we have

$$\Pr_{\mathcal{D}_U}(s) = |\phi|^2, \quad \text{with} \tag{S7}$$

$$\phi^* = \frac{1}{\sqrt{r!s!}} \sum_\sigma R_{\mathsf{in}_1,\mathsf{out}_{\sigma(1)}} R_{\mathsf{in}_2,\mathsf{out}_{\sigma(2)}} \dots R_{\mathsf{in}_n,\mathsf{out}_{\sigma(n)}} \tag{S8}$$

$$= \frac{1}{\sqrt{s!}} \sum_\sigma A_{1,\sigma(1)} A_{2,\sigma(2)} \dots A_{n,\sigma(n)}, \tag{S9}$$

where A is the $n \times n$ matrix formed by taking the appropriate number of copies of each row and column of $R_{m\times m}$. We have set $r! = 1$ since our input state has bosons in distinct sites. Continuing,

$$\Pr_{\mathcal{D}_U}(s) = \frac{1}{s!} \sum_\sigma |R_{\mathsf{in}_1,\mathsf{out}_{\sigma(1)}}|^2 |R_{\mathsf{in}_2,\mathsf{out}_{\sigma(2)}}|^2 \dots |R_{\mathsf{in}_n,\mathsf{out}_{\sigma(n)}}|^2 +$$
$$\frac{1}{s!} \sum_{\sigma\neq\tau} R_{\mathsf{in}_1,\mathsf{out}_{\sigma(1)}} R_{\mathsf{in}_2,\mathsf{out}_{\sigma(2)}} \dots R_{\mathsf{in}_n,\mathsf{out}_{\sigma(n)}} (R_{\mathsf{in}_1,\mathsf{out}_{\tau(1)}} R_{\mathsf{in}_2,\mathsf{out}_{\tau(2)}} \dots R_{\mathsf{in}_n,\mathsf{out}_{\tau(n)}})^*. \tag{S10}$$

The probability distribution $\mathcal{D}_{DP}$ that the algorithm samples from is given by the first line of Eq. (S10):

$$\Pr_{\mathcal{D}_{DP}}(s) = \frac{1}{s!} \sum_\sigma \mathcal{P}_{\mathsf{in}_1,\mathsf{out}_{\sigma(1)}} \mathcal{P}_{\mathsf{in}_2,\mathsf{out}_{\sigma(2)}} \dots \mathcal{P}_{\mathsf{in}_n,\mathsf{out}_{\sigma(n)}}, \tag{S11}$$

where the sum is over all the $n!$ ways of assigning the n input states to the n output states. As before, the $s!$ is to account for overcounting when two distinct permutations in the boson index refer to the same site index in the output state.

The expression for the probability is proportional to the permanent of the matrix with the positive entries $\mathcal{P}_{\mathsf{in}_i,\mathsf{out}_j}$, and can hence be efficiently approximated [S1]. Note that the algorithm does not explicitly calculate this probability but only samples from the distribution. We can now prove Lemma 3 of the main text.

Proof of Lemma 3. The variation distance is given by

$$\varepsilon = \sum_s \frac{1}{2s!}\left|\sum_{\sigma\neq\tau} R_{\mathsf{in}_1,\mathsf{out}_{\sigma(1)}}\dots R_{\mathsf{in}_n,\mathsf{out}_{\sigma(n)}}(R_{\mathsf{in}_1,\mathsf{out}_{\tau(1)}}\dots R_{\mathsf{in}_n,\mathsf{out}_{\tau(n)}})^*\right| \tag{S12}$$

$$\leq \sum_s \frac{1}{2}\sum_{\sigma\neq\tau}|R_{\mathsf{in}_1,\mathsf{out}_{\sigma(1)}}\dots R_{\mathsf{in}_n,\mathsf{out}_{\sigma(n)}}||R_{\mathsf{out}_{\tau(1)},\mathsf{in}_1}\dots R_{\mathsf{out}_{\tau(n)},\mathsf{in}_n}| \tag{S13}$$

$$= \sum_s \frac{1}{2}\sum_{\sigma,\rho}|R_{\mathsf{in}_1,\mathsf{out}_{\sigma(1)}}R_{\mathsf{out}_{\sigma(1)},\mathsf{in}_{\rho(1)}}|\dots|R_{\mathsf{in}_n,\mathsf{out}_{\sigma(n)}}R_{\mathsf{out}_{\sigma(n)},\mathsf{in}_{\rho(n)}}|, \tag{S14}$$

where $\rho=\tau^{-1}\circ\sigma\neq\mathsf{Id}$, the identity permutation. The last equality comes from rearranging the terms in $|R_{\mathsf{out}_{\tau(1)},\mathsf{in}_1}\dots R_{\mathsf{out}_{\tau(n)},\mathsf{in}_n}|$ so that the terms involving $R_{\mathsf{in}_i,\mathsf{out}_{\sigma(i)}}$ and $R_{\mathsf{out}_{\sigma(i)},j}$ (for some j) are together:

$$\sum_\sigma\sum_\tau\prod_i|R_{\mathsf{out}_{\tau(i)},\mathsf{in}_i}| = \sum_\sigma\sum_\tau\prod_i|R_{\mathsf{out}_i,\mathsf{in}_{\tau^{-1}(i)}}| \xrightarrow[i\to\sigma(i)]{\text{rearrange}} \sum_\sigma\sum_\tau\prod_i|R_{\mathsf{out}_{\sigma(i)},\mathsf{in}_{\tau^{-1}(\sigma(i))}}| \tag{S15}$$

Summing over all outcomes s (or configurations out), Eq. (S14) is equivalent to

$$\varepsilon \leq \frac{1}{2}\sum_j\sum_{\rho\neq\mathsf{Id}}\prod_i|R_{\mathsf{in}_i,j_i}||R_{j_i,\mathsf{in}_{\rho(i)}}|, \tag{S16}$$

where the sum j is over ordered tuples $(j_1,\dots j_n)$, representing the intermediate lattice sites that the bosons in positions $(\mathsf{in}_1,\dots\mathsf{in}_n)$ jump to, before jumping back to positions $(\mathsf{in}_{\rho(1)},\dots\mathsf{in}_{\rho(n)})$.

We can proceed to break the sum in Eq. (S16) based on the number of fixed points of the permutation ρ, that is, the number of indices i such that $\rho(i)=i$. We bound these quantities separately as follows:

$$\sum_{\substack{j_i,\rho\\ \rho(i)\neq i}}|R_{\mathsf{in}_i,j_i}||R_{j_i,\mathsf{in}_{\rho(i)}}| = C_i \leq c\ \forall\ i \quad \text{and}$$
$$\sum_{\substack{j_i,\rho\\ \rho(i)= i}}|R_{\mathsf{in}_i,j_i}||R_{j_i,\mathsf{in}_{\rho(i)}}| = \sum_{j_i}|R_{\mathsf{in}_i,j_i}|^2 = D_i = 1\ \forall\ i. \tag{S17}$$

The variation distance is therefore bounded above:

$$\varepsilon \leq \frac{1}{2}\sum\prod_{i\in\mathcal{I}_C}C_i\prod_{k\in\mathcal{I}_D}D_k, \tag{S18}$$

where the sum is over subsets $\mathcal{I}_C$ of the indices representing the input state, in. $\mathcal{I}_D$ is the complement of $\mathcal{I}_C$ and $|\mathcal{I}_D|$ is the number of fixed points. Suppose we find an upper bound c for C_i in Eq. (S17), we then have

$$\varepsilon \leq \frac{1}{2}\sum_{l=|\mathcal{I}_C|=2}^{n}\binom{n}{l}c^l = (c+1)^n - nc - 1. \tag{S19}$$

In Lemma 4, we show that $c=\eta L^{d-1}\mathrm{e}^{(vt-L)/\xi}$ for some constant η. Continuing from Eq. (S19),

$$\varepsilon \leq \frac{1}{2}[(c+1)^n - nc - 1] \tag{S20}$$

$$= \binom{n}{2}(1+h)^{n-2}c^2 \text{ for some } h\in[0,c] \text{ (by Taylor's theorem)} \tag{S21}$$

$$\varepsilon \leq \exp\left[2\log n + (n-2)\log(1+c) + 2\log c\right] \tag{S22}$$

Now, plugging in the value of c and assuming that $vt \le 0.9L$ and $\beta > 1$, we get

$$\varepsilon \le O\left(\exp\left[(n-2)\times \eta L^{d-1}\mathrm{e}^{(vt-L)/\xi} + 2\frac{vt-L}{\xi} + 2(d-1)\log L\right]\right) \tag{S23}$$

$$\le O\left(\exp\left[2\frac{vt-L}{\xi} + 2(d-1)\log L\right]\right). \tag{S24}$$

In the first line, we use the inequality $\log(1+x) \le x$ and in the second, $\exp\left[(n-2)\times \eta L^{d-1}\mathrm{e}^{(vt-L)/\xi}\right] = O(1)$ since $vt < L$ and $|vt - L| = \Omega(n^{\beta-1})$. This completes the proof of Lemma 3. □

Lemma 4. *For all constant dimensions d, $c = \eta L^{d-1}\mathrm{e}^{(vt-L)/\xi}$.*

Proof. Recall that

$$C_i = \sum_{j_i}\sum_{\rho(i)\neq i} |R_{\mathsf{in}_i,j_i}|\left|R_{j_i,\mathsf{in}_{\rho(i)}}\right|. \tag{S25}$$

Since we are looking for a bound that applies for all i, let us, for convenience, make the following changes in notation: $\mathsf{in}_i \to i, j_i \to j, \mathsf{in}_{\rho(i)} \to k$, denoting a boson starting at position i, jumping to j and then to k, where i and $k \neq i$ are site indices belonging to in. We split the sum in C_i based on the distance between i and j, $\ell_{ij} =: \ell$.

$$C_i = \sum_{\substack{j\\ \ell\le L}}\sum_{\substack{k\in\mathsf{in}\\ k\neq i}} |R_{i,j}|\,|R_{j,k}| + \sum_{\substack{j\\ \ell> L}}\sum_{\substack{k\in\mathsf{in}\\ k\neq i}} |R_{i,j}|\,|R_{j,k}|. \tag{S26}$$

Consider the first term:

$$\sum_{\substack{j\\ \ell\le L}} |R_{i,j}| \sum_{\substack{k\in\mathsf{in}\\ k\neq i}} |R_{j,k}| \le \sum_{\substack{j\\ \ell\le L}} |R_{i,j}| \sum_{\substack{k\in\mathsf{in}\\ k\neq i}} \mathrm{e}^{(vt-\ell_{kj})/\xi} \tag{S27}$$

$$\le \sqrt{\left(\sum_{\substack{j\\ \ell\le L}} 1^2\right)\left(\sum_{\substack{j\\ \ell\le L}} |R_{i,j}|^2\right)}\,\mathrm{e}^{vt/\xi}\sum_{k:\|\vec{x}_k\|\ge 1} \mathrm{e}^{(-2L\|\vec{x}_k\|+L)/\xi} \tag{S28}$$

$$\le aL^{d/2}e^{(vt+L)/\xi}\sum_{k:||\vec{x}_k||\ge 1} e^{-2L||\vec{x}_k||/\xi} \tag{S29}$$

$$\le abe^{(vt-L)/\xi}L^{d/2}. \tag{S30}$$

Here in the first line, we have used the Lieb-Robinson bound Eq. (2) of the main text. In the second line, we use the Cauchy-Schwarz inequality and the fact that $\ell = \ell_{ij} \le L$. In the second line, $\vec{x}_k$ is the position vector of site k relative to site i, re-scaled by $2L$. Therefore the sum over $\vec{x}_k$ is over all vectors with integer coordinates. In the last line, we use Lemma 6, to be proven below. a and b are constants independent of n that depend on the dimension d and the length scale ξ.

Now, in the second term for C_i in Eq. (S26), the intermediate site j is not necessarily close to i. Therefore, there are terms where j is close to $k \neq i$ and one has to treat these terms carefully. For these terms, we use the trivial Lieb-Robinson bound of 1 in Eq. (2) rather than $\exp\left(\frac{vt-\ell_{jk}}{\xi}\right) > 1$.

$$\sum_{\substack{j\\ \ell>L}}|R_{i,j}|\sum_{\substack{k\in\mathsf{in}\\ k\neq i}}|R_{j,k}| \leq \sum_{\substack{j\\ \ell>L}}|R_{i,j}|\sum_{\substack{k\in\mathsf{in}\\ k\neq i}}\min(1,\mathrm{e}^{(vt-\ell_{k,j})/\xi}) \tag{S31}$$

$$\leq \sum_{\substack{j\\ \ell>L}}|R_{i,j}|\left(1+\mathrm{e}^{vt/\xi}\sum_{k:||\vec{x}_k||\geq 1}\mathrm{e}^{(L-2L\|\vec{x}_k\|)/\xi}\right) \tag{S32}$$

$$\leq \left(1+b\mathrm{e}^{(vt-L)/\xi}\right)\sum_{\substack{j\\ \ell>L}}|R_{i,j}| \tag{S33}$$

$$\leq \left(1+b\mathrm{e}^{(vt-L)/\xi}\right)\mathrm{e}^{vt/\xi}\sum_{\substack{j\\ \ell>L}}\mathrm{e}^{-\ell/\xi} \tag{S34}$$

$$\leq \left(1+b\mathrm{e}^{(vt-L)/\xi}\right)\tilde{b}L^{d-1}\mathrm{e}^{(vt-L)/\xi}. \tag{S35}$$

In the second line, we use 1 as a Lieb-Robinson bound for $|R_{j,k}|$ when $k=k^*$, the site belonging to in that is closest to j. All other k's have distances from j bounded below by $2L\|\vec{x}_k\|-L$, where $\vec{x}_k$ is now the re-scaled position vector of a site k with the origin at k^*. We apply Lemma 6 in the third line and Lemma 5 in the fifth. Collecting everything, we have

$$C_i \leq ab\mathrm{e}^{(vt-L)/\xi}L^{d/2}+\left(1+b\mathrm{e}^{(vt-L)/\xi}\right)\tilde{b}L^{d-1}\mathrm{e}^{(vt-L)/\xi} \tag{S36}$$

$$\leq \mathrm{e}^{(vt-L)/\xi}\eta L^{d-1} \text{ for large enough } L. \tag{S37}$$

□

Lemma 5 (d-dimensional sum)**.** *The sum $\sum_{\|\vec{x}\|\geq L}\mathrm{e}^{-\|\vec{x}\|/\xi}$ over points $\vec{x}$ with integer coordinates is upper bounded by $a\mathrm{e}^{-L/\xi}\left(\xi L^{d-1}\right)$ for large enough $\frac{L}{\xi}$ for some dimension-dependent constant a_d.*

Proof. We can view the sum over a lattice of vectors with integer coordinates as a Riemann sum and bound the corresponding d-dimensional integral. Consider the quantity

$$g=\int_{\|\vec{x}\|\geq L}\mathrm{e}^{-\|\vec{x}\|/\xi}\mathrm{d}^d\vec{x}=\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\xi^d\Gamma\left(d,\frac{L}{\xi}\right), \tag{S38}$$

where $\Gamma(d,x)=\int_x^\infty y^{d-1}\mathrm{e}^{-y}\mathrm{d}y$ is the incomplete Γ function. We can lower bound the integral by the Riemann sum $\sum_\Delta V_\Delta \mathrm{e}^{-\|\vec{y}\|/\xi}$, where the sum is over cells Δ with volume V_Δ centered at lattice points $\vec{x}$. $\vec{y}$ is the point in the cell Δ with the highest norm $\|\vec{y}\|$. Further, the point with the highest norm is not too distant from the one at the center: $\|\vec{y}\|\leq\|\vec{x}\|+\frac{\sqrt{d}}{2}$. Therefore, we have

$$f:=\sum_{\|\vec{x}\|\geq L}\mathrm{e}^{\|\vec{x}\|/\xi}\leq g\times\mathrm{e}^{\sqrt{d}/(2\xi)}. \tag{S39}$$

We now need an upper bound on the incomplete Γ function $\Gamma(d,x)$ for large x [S2]:

$$\Gamma(d,x)\to x^{d-1}\mathrm{e}^{-x}\left(1+O\left(\frac{1}{x}\right)\right) \text{ as } x\to\infty. \tag{S40}$$

Combining Eq. (S38) and Eq. (S39), we get:

$$f\leq O\left(\xi L^{d-1}\exp\left[-\frac{L}{\xi}\right]\right). \tag{S41}$$

□

Using a similar method, we also get bounds on a related sum.

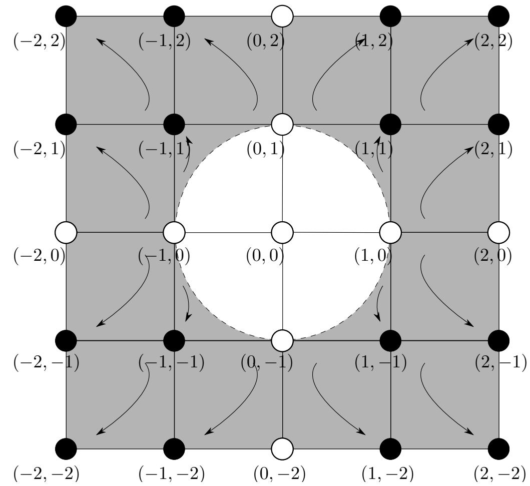

FIG. 2. Part of the lattice of vectors with integer coordinates. The black dots are the points in the cell with the maximum norm $\|\vec{y}\|$ and the exponential is evaluated at these points. The white ones do not enter the Riemann sum and are related to f_{d-1}, the corresponding quantity in one lower dimension. The arrows show which point in the cell is picked to lower bound the Riemann sum.

Lemma 6. *For $\vec{x} \in \mathbb{Z}^d$, $f_d := \sum_{\|\vec{x}\|\geq 1} \mathrm{e}^{-2L\|\vec{x}\|/\xi} \leq b_d \exp[-\frac{2L}{\xi}]$ for some dimension-dependent constant b_d.*

Proof. We prove the statement by induction on the dimension d. For $d = 1$, the statement is seen to be true since the sum evaluates exactly:

$$f_1 = \sum_{|x|\geq 1} \mathrm{e}^{-2L|x|/\xi} = 2\sum_{x=1}^{\infty} \mathrm{e}^{-2Lx/\xi} = \frac{2\mathrm{e}^{-2L/\xi}}{1-\mathrm{e}^{-2L/\xi}} \leq 2.1 \times \mathrm{e}^{-2L/\xi}. \tag{S42}$$

For the inductive step, consider the integral $g(d) = \int_{\|\vec{y}\|\geq 1}^{\infty} \mathrm{e}^{-2L\|\vec{y}\|/\xi} \mathrm{d}^d\vec{y}$. This is lower bounded by the Riemann sum represented in Fig. 2. The white dots represent vectors with at least one zero coordinate and do not enter the Riemann sum according to this way of dividing the region of integration into cells. In the following, the set of points with at least one zero coordinate is denoted $\emptyset_{cc}$. We have:

$$g(d) = \int_{\|\vec{y}\|\geq 1}^{\infty} \mathrm{e}^{-2L\|\vec{y}\|/\xi} \mathrm{d}^d\vec{y} \geq \sum_{\vec{x}\notin\emptyset_{cc}} \mathrm{e}^{-2L\|\vec{x}\|/\xi} \Delta_{\vec{x}}, \tag{S43}$$

where $\Delta_{\vec{x}}$ is the volume of the cell associated with the lattice vector $\vec{x}$. In Fig. 2, the volume of most cells (whose center is at distance 1.5 or beyond from the origin) is 1. The cells near the origin have some volume $\alpha_d < 1$ that depends on the dimension. Lower bounding all volumes $\Delta_{\vec{x}}$ by α_d,

$$\sum_{\vec{x}\notin\emptyset_{cc}} \mathrm{e}^{-2L\|\vec{x}\|/\xi} < \frac{g(d)}{\alpha_d} \tag{S44}$$

$$= \frac{2\pi^{d/2}}{\alpha_d \Gamma(\frac{d}{2})} \left(\frac{\xi}{2L}\right)^d \Gamma\left(d, \frac{2L}{\xi}\right). \tag{S45}$$

Now it remains to upper bound the contribution from summing over the points $\emptyset_{cc}$. Notice that the sum over these points is upper bounded by the sum over d hyperplanes of dimenson $d-1$. From the inductive hypothesis,

$$\sum_{\vec{x}\in\emptyset_{cc}} \mathrm{e}^{-2L\|\vec{x}\|/\xi} \leq d f_{d-1} \tag{S46}$$

since there are d hyperplanes of dimension $d-1$. Adding Eqs. (S45) and (S46), we get

$$\sum_{\vec{x}} \mathrm{e}^{-2L\|\vec{x}\|/\xi} = f_d < df_{d-1} + \frac{2\pi^{d/2}}{\alpha_d \Gamma(\frac{d}{2})} \left(\frac{\xi}{2L}\right)^d \Gamma\left(d, \frac{2L}{\xi}\right). \tag{S47}$$

$$\leq db_{d-1} \exp\left[-\frac{2L}{\xi}\right] + \frac{2\pi^{d/2}}{\alpha_d \Gamma(\frac{d}{2})} \left(\frac{\xi}{2L}\right) \exp\left[-\frac{2L}{\xi}\right] \tag{S48}$$

$$f_d < b_d \exp\left[-\frac{2L}{\xi}\right], \tag{S49}$$

proving the lemma. In the second line we have expanded the incomplete Γ function for large L/ξ. □

[S1] M. Jerrum, A. Sinclair, and E. Vigoda, J. ACM **51**, 671 (2004).
[S2] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, "NIST Digital Library of Mathematical Functions," .